Problem Set 5 Solutions

Problem 5-1. Let $G = (V, E)$ be an undirected graph. Define the *line graph* $G' = (E, E')$ as the graph whose vertices are the edges of G, and $(e_1, e_2) \in E'$ iff e_1 and e_2 share a vertex in G.

Theorem 1 *If G is Eulerian then so is G'.*

Proof. We first show that if G is connected then so is G'. Pick any two vertices e_s, e_t of G'. Let $e_s = (u, v), e_t = (x, y)$ in G, where $u, v, x, y \in V$. Since G is connected, there is a path from u to x. Let the path be $u \equiv v_0 \to v_1 \to v_2 \cdots \to v_{k-1} \to x \equiv v_k$. Edges on the path are $e_{i_1} e_{i_2} \cdots e_{i_k}$ where $e_{i_j} = (v_{j-1}, v_j)$, $j = 1, 2, \cdots k$. Then $e_s \equiv e_{i_0} \to e_{i_1} \to e_{i_2} \cdots \to e_{i_k} \to e_{i_{k+1}} \equiv e_t$ is a path from e_s to e_t in G', since for $0 \leq j \leq k$, edge e_{i_j} and $e_{i_{j+1}}$ share an endpoint v_j.

We next show that if all vertices in G have even degree then so does each vertex of G'. Take any vertex e of G'. Let $e = (u, v)$. Define $N(i)$ = set of neighbors of vertex i, and $d(i) = |N(i)|$ is the degree of node i. If $e_i \in N(e)$ in graph G', then e_i and e share an endpoint in G. That is, $e_i = (u, w)$ or $e_i = (v, w)$, $w \in N(u) \cup N(v)$, where $w \neq u$ or v. So $d(e) = |N(e)| = |N(u)| + |N(v)| - 2 = d(u) + d(v) - 2$ is even, if $d(u), d(v)$ are even.

Finally, if G is Eulerian, then G is connected and has only even degree vertices, therefore G' is connected and has only even degree vertices. Therefore G' is Eulerian. ∎

Problem 5-2. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two trees on the same set of vertices. Define the graph $G = (V, E)$ where $E = E_1 \cup E_2$.

Theorem 2 *G is 4-colorable.*

Proof. First 2-color G_1 and G_2 seperately. This is possible since we know that all trees are 2-colorable. Then for any vertex $v \in V$, color it (c_1, c_2) in graph G if it is colored c_1 in G_1 and c_2 in G_2. Clearly, (c_1, c_2) is of size $2 \times 2 = 4$. Now prove that this is a legal coloring. For any edge $(u, v) \in E$, say we get color (c_1, c_2) and (c_1', c_2') by the above scheme. Since $E = E_1 \cup E_2$, we have either $(u, v) \in E_1$ or $(u, v) \in E_2$. WLOG, say $(u, v) \in E_1$. c_1, c_1' is a 2-coloring for E_1, therefore, $c_1 \neq c_1'$. So $(c_1, c_2) \neq (c_1', c_2')$. ∎

It is easy to see that this theorem is optimal with respect to the number of colors used: Figure 1 depicts the graph K_4 (the complete graph on 4 vertices), which can be easily decomposed into two trees but cannot be colored with fewer than 4 colors.

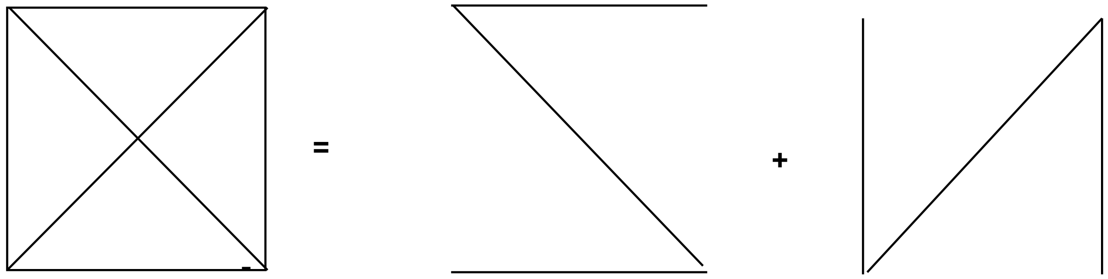

Figure 1: An example of a non 3-colorable graph that is union of two trees.

Problem 5-3. **Theorem 3** *$\{\bar{\wedge}\}$ (NAND) is a complete basis but $\{\rightarrow\}$ is not.*

Proof.

(a) $\{\bar{\wedge}\}$ is a complete basis. We show how to implement $\neg$, $\wedge$, and $\vee$ with $\bar{\wedge}$:

- $\neg A = \neg(A \wedge A) = A \bar{\wedge} A$
- $A \wedge B = \neg(\neg(A \wedge B)) = \neg(A \bar{\wedge} B) = (A \bar{\wedge} B) \bar{\wedge} (A \bar{\wedge} B)$
- $A \vee B = \neg(\neg A \wedge \neg B)) = \neg A \bar{\wedge} \neg B = (A \bar{\wedge} A) \bar{\wedge} (B \bar{\wedge} B)$

Since $\{\neg, \wedge, \vee\}$ is a complete basis, so is $\{\bar{\wedge}\}$.

(b) $\{\rightarrow\}$ is not a complete basis since it can not simulate $\neg$. Consider a circuit of $\rightarrow$-gates. At the bottom level we have only A's as input. If $A = T$, then each gate at the first level outputs T. By induction on the levels, there is no way we can get F as output from this $\rightarrow$ circuit.

$\blacksquare$

Problem 5-4. For x and $y \in \{0,1\}^n$ we say that $x \leq y$ if $x_i \leq y_i$ for all i. A function $f : \{0,1\}^n \rightarrow \{0,1\}$ is *monotone* if $x \leq y$ implies that $f(x) \leq f(y)$. Another way of saying this is that if $f(z) = 1$ then changing any bit that is 0 in z to 1 should not change the value of the function from 1 to 0.

If $g : \{0,1\}^k \rightarrow \{0,1\}$ and $f : \{0,1\}^n \rightarrow \{0,1\}$ are boolean functions we define the composition of f and g as $(f \circ g) : \{0,1\}^{nk} \rightarrow \{0,1\}$ where $(f \circ g)(x_1 x_2 \dots x_n) = f(g(x_1)g(x_2)\dots g(x_n))$.

Theorem 4 *The conjunction, disjunction, and composition of monotone functions are monotone.*

Proof. Let $x \leq y$. If f and g are monotone, then $f(x) \leq f(y)$ and $g(x) \leq g(y)$.

(a) Show $(f \wedge g)(x) = f(x) \wedge g(x)$ is monotone:
$f(x) \wedge g(x) = \min(f(x), g(x)) \leq \min(f(y), g(y)) = f(y) \wedge g(y)$. Therefore, $(f \wedge g)(x) \leq (f \wedge g)(y)$. So $(f \wedge g)$ is monotone.

(b) Show $(f \vee g)(x) = f(x) \vee g(x)$ is monotone:
$f(x) \vee g(x) = \max(f(x), g(x)) \leq \max(f(y), g(y)) = f(y) \vee g(y)$. Therefore, $(f \vee g)(x) \leq (f \wedge g)(y)$. So $(f \vee g)$ is monotone.

(c) Show $(f \circ g)(x) = f(g(x_1)g(x_2)\dots g(x_n))$ is monotone:
g is monotone implies that $g(x_i) \leq g(y_i)$ for all $1 \leq i \leq n$. Let $x' = g(x_1)g(x_2)\dots g(x_n)$ and $y' = g(y_1)g(y_2)\dots g(y_n)$, then $x' \leq y'$. So $f(x') \leq f(y')$, since f is monotone. That is $(f \circ g)(x) \leq (f \circ g)(y)$. So $(f \circ g)$ is monotone.

■

Problem 5-5. Prove that $\{A \to (B \wedge D), B \to D\} \vdash A \to D$.

Proof. Let $\Lambda = \{A \to (B \wedge D), B \to D\}$.

1. $\Lambda \vdash A \to (B \wedge D)$	$\psi \in \Gamma$.
2. $\Lambda \cup \{A\} \vdash A \to (B \wedge D)$	from rule 10 and step 1.
3. $\Lambda \cup \{A\} \vdash A$	$\psi \in \Gamma$.
4. $\Lambda \cup \{A\} \vdash B \wedge D$	from rule 4 and step 2 and 3.
5. $\Lambda \cup \{A\} \vdash D \wedge B$	from rule 11, $D \wedge B = B \wedge D$ and step 4.
6. $\Lambda \cup \{A\} \vdash D$	from rule 6 and step 5.
7. $\Lambda \vdash A \to D$	from rule 3 and step 6.

■