Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 15
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Problem Set 6

Due: In recitation, April 7.

Problem 6-1. Give an example of two functions f and g that are incomparable using asymptotic notation. (i.e. $f(n)$ is not $O(g(n))$, $g(n)$ is not $O(f(n))$, $f(n)$ is not $\Omega(g(n))$, and so on)

Problem 6-2. Rank the following functions by order of growth; that is find an arrangement $g_1, g_2, \ldots, g_{30}$ of the functions satisfying $g_1 = \Omega(g_2), g_2 = \Omega(g_3), \ldots, g_{29} = \Omega(g_{30})$. Partition your list into equivalence classes such that $f(n)$ and $g(n)$ are in the same class iff $f(n) = \Theta(g(n))$.

$\lg(\lg^* n)$	$2^{\lg^* n}$	$(\sqrt{2})^{\lg n}$	n^2	$n!$	$(\lg n)!$
$(3/2)^n$	n^3	$\lg^2 n$	$\lg(n!)$	2^{2^n}	$n^{(1/\lg n)}$
$\ln\ln n$	$\lg^* n$	$n2^n$	$n^{\lg\lg n}$	$\ln n$	1
$2^{\lg n}$	$(\lg n)^{\lg n}$	e^n	$4^{\lg n}$	$(n+1)!$	$\sqrt{\lg n}$
$\lg^*(\lg n)$	$2^{\sqrt{2\lg n}}$	n	2^n	$n\lg n$	$2^{2^{n+1}}$

Here's a definition for $\lg^* n$ ("log star of n"). We first define the function $\lg^{(i)} n$.

$$\lg^{(i)} n = \begin{cases} n & \text{if } i = 0, \\ \lg(\lg^{(i-1)} n) & \text{if } i > 0 \text{ and } \lg^{(i-1)} n > 0, \\ \text{undefined} & \text{if } i > 0 \text{ and } \lg^{(i-1)} n \leq 0 \text{ or } \lg^{(i-1)} n \text{ is undefined} \end{cases}$$

Now $\lg^* n = \min\{i \geq 0 : \lg^{(i)} n \leq 1\}$. For example $\lg^* 2 = 1, \lg^* 4 = 2, \lg^* 16 = 3, \lg^* 65536 = 4, \lg^*(2^{65536}) = 5$.

Problem 6-3. Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove or disprove each of the following conjectures.

(a) $f(n) = O(g(n))$ implies $g(n) = O(f(n))$.

(b) $f(n) + g(n) = \Theta(\min(f(n), g(n)))$.

(c) $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$.

(d) $f(n) = O((f(n))^2)$.

(e) $f(n) + o(f(n)) = \Theta(f(n))$.

Problem 6-4. Prove that:

$$\binom{n}{n/2} = \Theta(2^n/\sqrt{n})$$

Please be careful when you're manipulating the Θs. Remember that you have to prove both an upper and a lower bound.

Problem 6-5. Give asymptotic upper and lower bounds for $T(n)$ in the following recurrence.

$$T(n) = 2T(n/2) + n^3.$$