Massachusetts Institute of Technology | Handout 13
6.042J/18.062J: Mathematics for Computer Science | March 16,1995
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Problem Set 5

Due: In recitation, March 24.

Problem 5-1. Let $G = (V, E)$ be an undirected graph. Define the *line graph* $G' = (E, E')$ as the graph whose vertices are the edges of G, and $(e_1, e_2) \in E'$ iff e_1 and e_2 share an endpoint in G.

Argue that if G is Eulerian then so is G'. (Be sure to argue that if G is connected then so is G' as part of your proof).

Problem 5-2. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two trees on the same set of vertices. Define the graph $G = (V, E)$ where $E = E_1 \cup E_2$.

- **(a)** Prove that G is 4-colorable.
- **(b)** Give an example to show that the result cannot be improved. That is, show that there are 2 trees whose union is not 3-colorable.

Problem 5-3. Prove that $\{\bar{\wedge}\}$ (NAND) defined by $A \bar{\wedge} B = \neg(A \wedge B)$ is a complete basis but $\{\rightarrow\}$ is not.

Problem 5-4. For x and $y \in \{0,1\}^n$ we say that $x \leq y$ if $x_i \leq y_i$ for all i. A function $f : \{0,1\}^n \rightarrow \{0,1\}$ is *monotone* if $x \leq y$ implies that $f(x) \leq f(y)$. Another way of saying this is that if $f(z) = 1$ then changing any bit that is 0 in z to 1 should not change the value of the function from 1 to 0.

If $g : \{0,1\}^k \rightarrow \{0,1\}$ and $f : \{0,1\}^n \rightarrow \{0,1\}$ are boolean functions we define the composition of f and g as $(f \circ g) : \{0,1\}^{nk} \rightarrow \{0,1\}$ where $(f \circ g)(x_1 x_2 \ldots x_n) = f(g(x_1)g(x_2)\ldots g(x_n))$.

Prove that the conjunction, disjunction, and composition of monotone functions are monotone.

Problem 5-5. Prove that $\{A \rightarrow (B \wedge D), B \rightarrow D\} \vdash A \rightarrow D$.