Massachusetts Institute of Technology — Handout 10
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Problem Set 3 Solutions

Problem 3-1. Let n be a natural number greater than 2. Define relation R on $\mathbb{Z}$ by

$$a\,R\,b \iff n \text{ divides } a-b.$$

(a) R is an equivalence relation.

Proof.

1. **reflexivity:** $a\,R\,a$ because n divides $a-a=0$ for any a (since $0n=0$).
2. **symmetry:** If $a\,R\,b$, then $n \mid (a-b)$; that is, $a-b=kn$ for some integer k. Therefore, $n \mid (b-a)$ since $b-a=-kn$. Therefore $b\,R\,a$.
3. **transitivity:** If $a\,R\,b$ and $b\,R\,c$, then, $a-b=k_1n$ and $b-c=k_2n$ for $k_1,k_2\in\mathbb{Z}$. So $n \mid (a-c)$, since $a-c=(a-b)+(b-c)=(k_1+k_2)n$. Therefore $a\,R\,c$.

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(b) There are total n equivalence classes, namely

$$\{kn \mid k\in\mathbb{Z}\},\quad \{kn+1 \mid k\in\mathbb{Z}\},\quad \cdots\quad \{kn+(n-1) \mid k\in\mathbb{Z}\}$$

Problem 3-2. The following relation $\prec$ defines a partial order on $\{2,3,4,\cdots,100\}$:

$$a \prec b \iff a \text{ divides } b.$$

(a) The minimal elements are all the primes between 2 and 100. The maximal elements are all the numbers between 51 and 100.

(b) One longest chain is $\{2,\ 4,\ 8,\ 16,\ 32,\ 64\}$ (there are others of length 6).

(c) The partition

$$\{2,3\}\,\{4,5,6\}\,\{7,\cdots,12\}\,\{13,\cdots,25\}\,\{26,\cdots,50\}\,\{51,\cdots,100\}$$

is an antichain cover of size 6. Clearly we cannot have an antichain cover of size less than 6 since each the elements of the chain $\{2,\ 4,\ 8,\ 16,\ 32,\ 64\}$ must be put in a different antichain.

Problem 3-3.

Definition 1 *A partial ordering $\preceq$ of a set X is an* extension *of a partial ordering $\leq$ if*

$$x \leq y \Rightarrow x \preceq y$$

for all $x, y \in X$,

In other words, thinking of the relations $\preceq$ and $\leq$ as subsets of $X \times X$, we have $\leq \subseteq \preceq$.

Theorem 1 *Any partial ordering of a finite set can be extended to a total ordering of that set.*

Proof. By induction on the size of the set. In the base case, we have a set of size 1, which by itself is trivially a total order.

Now assume that any partial ordering on a finite set of size n can be extended to a total ordering on that set. Consider a partial ordering $\leq$ on set X of size $n+1$. Let m be a maximal element of the poset (one must exist; why?). Now $\leq$ induces a partial ordering of $X \setminus \{m\}$ in the obvious way. By our inductive hypothesis, we can extend this partial ordering on $X \setminus \{m\}$ to a total ordering $\preceq$ on $X \setminus \{m\}$. To obtain the desired total ordering of X, simply put m at the top of the chain given by $\preceq$ on $X \setminus \{m\}$ (that is, make m greater than every other element of X).

Clearly, this is a total ordering. It is an extension of $\leq$ because the ordering for elements in $X \setminus \{m\}$ is preserved according to our inductive hypothesis, and since m had no elements above it, orderings with respect to m are also preserved. Formally, if $x \leq m$ then $x \preceq m$ and if $x \leq y$, $y \neq m$ ($x \neq m$ because m is maximal), then $x \preceq y$ because by the inductive hypothesis $\preceq$ is an extension of the partial ordering on $X \setminus \{m\}$ induced by $\leq$. ∎

Problem 3-4. Let $\prec$ be a partial ordering. Define R by $b\,R\,a \iff a \prec b$.

R is also a partial ordering.

Proof.

(a) reflexivity: $a \prec a$ since $\prec$ is reflexive, so $a\,R\,a$.

(b) antisymmetry: If $a\,R\,b$ and $b\,R\,a$, then by definition of R, $b \prec a$ and $a \prec b$. Since $\prec$ is antisymmetric, we have $b = a$ as desired.

(c) transitivity: If $a\,R\,b$ and $b\,R\,c$, then by definition of R, $b \prec a$ and $c \prec b$. Since $\prec$ is transitive, we have $c \prec a$, which implies $a\,R\,c$.

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Notice that the partial order R is simply the partial order $\prec$ "upside-down".