Problem Set 4

Due: In recitation, March 10.

Problem 4-1. Prove that any countable cover of an uncountable set contains an uncountable set. (Recall that a *cover* of X is a collection of sets whose union is X).

Problem 4-2. Let A and B be finite sets. How many functions are there from A to B? That is, what is the cardinality of $\{f \mid f : A \to B\}$?

Problem 4-3. A certain hotel has 90 rooms and 100 guests. We wish to give keys to the guests so that any subset of 90 guests collectively hold keys to all of the rooms. (Different rooms require different keys.) What is the smallest number of keys that we have to distribute? Prove your answer.

Problem 4-4. Recall that the composition $g \circ f$ of two functions f and g is defined by $(g \circ f)(x) = g(f(x))$.

(a) Show that if $f : A \to B$ and $g : B \to C$ are both surjections, then $(g \circ f) : A \to C$ is also a surjection.

(b) Show that if $f : A \to B$ and $g : B \to C$ are both bijections, then $(g \circ f) : A \to C$ is also a bijection.

(c) If $g \circ f$ is a bijection, what can you conclude about f and g?

Problem 4-5. For any rooted binary tree, let $d(x)$ denote the depth of node x in the tree. (The depth of the root is 0.) Prove that

$$\sum_{\text{leaves } x} 2^{-d(x)} \leq 1$$

where the sum is over all *leaves* in the tree.