Problem Set 2 Solutions

Problem 2-1. Define the composition $P \circ Q$ of two relations as follows:
If $Q \subseteq A \times B$ and $P \subseteq B \times C$ then $P \circ Q = \{(a, c) : \exists b \text{ such that } (a, b) \in Q \text{ and } (b, c) \in P\}$.
Note that $P \circ Q \subseteq A \times C$.

Let $R \subseteq S \times S$ be a relation. Prove or disprove each of the following statements.

(a) If R is transitive, then $R \circ R$ is transitive.

Proof. Let $(a, b) \in R \circ R$ and $(b, c) \in R \circ R$. We want to show that (a, c) is in $R \circ R$, so we need to find an x such that (a, x) and (x, c) are both in R.

Since $(a, b) \in R \circ R$, there exists a d such that (a, d) and (d, b) are in R. Since R is transitive, it follows that $(a, b) \in R$. Similarly, there exists an e such that (b, e) and (e, c) are in R, and so $(b, c) \in R$. Taking $x = b$, we are done. ■

(b) If R is transitive, then $R \circ R \subseteq R$.

Proof. Let $(a, c) \in R \circ R$. Then there exists b such that (a, b) and (b, c) are both in R. Since R is transitive $(a, c) \in R$. Therefore $R \circ R \subseteq R$. ■

(c) If $R \circ R \subseteq R$, then R is transitive.

Proof. Let $(a, b), (b, c) \in R$. Then $(a, c) \in R \circ R$. Since $R \circ R \subseteq R$, $(a, c) \in R$ and so R is transitive.

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Problem 2-2. Prove or disprove the following identities:

(a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof. To prove this identity we need to show the following two containments:

1. $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$. Let $(x, y) \in A \times (B \cap C)$. Then $(x, y) \in A \times B$ since $x \in A$ and $y \in B$. Similarly $(x, y) \in A \times C$. Thus $(x, y) \in (A \times B) \cap (A \times C)$.
2. $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. Let $(x, y) \in (A \times B) \cap (A \times C)$. Then $x \in A$ and $y \in B \cap C$. Therefore $(x, y) \in A \times (B \cap C)$.

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(b) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$. This is false. To prove this it is enough to give an example where the equality above doesn't hold. Let $A = \{1\}, B = \emptyset, C = \emptyset, D = \{1\}$. Then $(A \cup B) \times (C \cup D) = \{(1, 1)\}$ but both $A \times C$ and $B \times D$ are empty.

(c) $(A - B) \times (C - D) = (A \times C) - (B \times D)$. Again this is false. A counterexample is: Let $A = \{1, 2\}, B = \{2\}, C = D = \{1\}$. Then $(A - B) \times (C - D) = \varnothing$ since $C - D = \varnothing$. But $A \times C = \{(1, 2), (2, 1)\}$ and $B \times D = \{(2, 1)\}$. Therefore $(A \times C) - (B \times D) = \{(1, 2)\} \neq \varnothing$.

(d) $(A \triangle B) \times (C \triangle D) = (A \times C) \triangle (B \times D)$. A counterexample is: Let $A = \varnothing, B = \{1\}, C = \{1\}, D = \varnothing$. Then $A \triangle B = \{1\}, C \triangle D = \{1\}$ and so $(A \triangle B) \times (C \triangle D) = \{(1, 1)\}$. However both $A \times C$ and $B \times D$ are empty and so $(A \times C) \triangle (B \times D) = \varnothing$.

Problem 2-3. Let $f^{(i)}(x)$ denote the ith derivative of $f(x)$. Prove that if $f(x)$ and $g(x)$ are real-valued functions of one variable with derivatives of all orders, then

$$(f(x)g(x))^{(n)} = \sum_{j=0}^{n} \binom{n}{j} f^{(n-j)}(x) g^{(j)}(x)$$

Proof. We will proceed by induction on n.

Base Case: $n = 1$. $(f(x)g(x))^{(1)} = f^{(1)}(x)g^{(0)}(x) + f^{(0)}(x)g^{(1)}(x) = \sum_{j=0}^{1} f^{(n-j)}(x)g^{(j)}(x)$.

Inductive Step: Assume that:

$$(f(x)g(x))^{(n)} = \sum_{j=0}^{n} \binom{n}{j} f^{(n-j)}(x) g^{(j)}(x)$$

Then $(fg)^{(n+1)} = ((fg)^{(n)})'$. Thus:

$$\begin{aligned}
(f(x)g(x))^{(n+1)} &= \sum_{j=0}^{n} \binom{n}{j} \left[f^{(n-j+1)}(x) g^{(j)}(x) + f^{(n-j)}(x) g^{(j+1)}(x) \right] \\
&= \sum_{j=0}^{n} \binom{n}{j} f^{(n+1-j)}(x) g^{(j)}(x) + \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(n+1-k)}(x) g^{(k)}(x) \\
&\quad (k = j + 1 \text{ in the second summation}) \\
&= \sum_{j=0}^{n} \left[\binom{n}{j} + \binom{n}{j-1} \right] f^{(n+1-j)}(x) g^{(j)}(x) \\
&= \sum_{j=0}^{n+1} \binom{n+1}{j} f^{(n+1-j)}(x) g^{(j)}(x) \\
&\quad \text{(from PS1.3)}
\end{aligned}$$

which is the desired identity for $n + 1$.

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Problem 2-4. You are given an unlimited supply of 4-cent and 7-cent stamps. What is the largest amount of postage that you cannot make with these two denominations? Prove your answer.

Some preliminary observations:

1. We can't make 17 cents. Why? It's impossible to do it using 0, 1, or 2 7-cent stamps.
2. We can make 18, 19, 20, and 21 cents.
   - $18 = 2 \times 7 + 1 \times 4$
   - $19 = 1 \times 7 + 3 \times 4$
   - $20 = 0 \times 7 + 5 \times 4$
   - $21 = 3 \times 7 + 0 \times 4$

We now prove that 17 is the largest amount of postage we can't make.

Theorem 1 *If $n \geq 18$ then we can make n cents with 4-cent and 7-cent stamps.*

Proof. We go by induction on n.

Base Case: $n = 18$. See Observation 2.

Inductive Step: Assume that if $18 \leq k < n$ then we can make k cents with 4-cent and 7-cent stamps. We have two cases:

1. If $n - 4 < 18$, then n is on the list in Observation 2.
2. If $n - 4 \geq 18$, then we use the inductive hypothesis to make $n - 4$ cents and then make n cents by using an extra 4-cent stamp.

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