# Problem Set 1 Solutions

**Problem 1-1.** Let $F_i$ be the $ith$ Fibonacci number; that is,

$$F_0 = 0,\ F_1 = 1, \quad \text{and} \quad F_i = F_{i-1} + F_{i-2}, \quad \text{for } i \geq 2.$$

Show that $1 + \sum_{i=0}^{k} F_i = F_{k+2}$ for all $k \geq 0$.

*Proof.*

By induction on $k$.

1. **Base case:** $k = 0$.

$$F_{k+2} = F_2 = F_1 + F_0 = 1 + 0 = 1, \text{ and } 1 + \sum_{i=0}^{k} F_i = 1 + F_0 = 1 + 0 = 1.$$

   Therefore, the claim is true for $k = 0$.

2. **Inductive Step:**

   Show that for any positive integer $k$, if the claim is true for $k$ then the claim must also be true for $k + 1$.

   Fix any $k \geq 0$. If the claim is true for $k$, then $1 + \sum_{i=0}^{k} F_i = F_{k+2}$.

   We show that the claim for $k + 1$ must therefore also be true:

$$1 + \sum_{i=0}^{k+1} F_i = 1 + \sum_{i=0}^{k} F_i + F_{k+1} = F_{k+2} + F_{k+1} = F_{k+3} = F_{(k+1)+2}$$

   Here, the second equality follows from the inductive hypothesis, and the third equality from the definition of Fibonacci numbers.

Therefore, by induction, $1 + \sum_{i=0}^{k} F_i = F_{k+2}$ for all integers $k \geq 0$. ∎

**Problem 1-2.** Prove that for each integer $n > 1$, the expression

$$\underbrace{\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}_{n\text{ 1's}}$$

is irrational. For example, for $n = 2$ this is $\sqrt{1+\sqrt{1}}$.

*Proof.* By induction on $n$. To simplify notation, let us define

$$a_n \equiv \underbrace{\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}_{n\text{ 1's}}.$$

So the problem is to prove that $a_n$ is irrational for all integers $n > 1$.

1. **Base case:** $n = 2$.
   We showed in class that $a_2 = \sqrt{1+1} = \sqrt{2}$ is irrational.
2. **Inductive Step:**
   Show that for any integer $n \geq 2$, if the claim is true for $n$ then the claim must also be true for $n + 1$.
   If the claim is true for $n$, then $a_n$ is irrational. We must prove that $a_{n+1}$ is also irrational. We prove this by contradiction. If $a_{n+1}$ is a rational number, then $a_{n+1} = p/q$ for some integers $p$ and $q$. Since $a_{n+1} = \sqrt{1 + a_n}$, we have $a_n = p^2/q^2 - 1 = (p^2 - q^2)/q^2$, a rational number, which is a contradiction. We conclude that $a_{n+1}$ must be irrational.

Therefore, by induction, $a_n$ is irrational for all integers $n > 1$. ■

**Problem 1-3.** Define the binomial coefficient $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ for $0 \leq k \leq n$. $0!$ is defined to be 1. The binomial coefficient is defined to be 0 if $k < 0$ or $k > n$.

**(a)** For any integers $n$ and $k$ such that $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

*Proof.* We may verify this directly using the formula given. As mentioned in class, the equality holds only for $n > 0$.

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!\,(n-1-k)!} + \frac{(n-1)!}{(k-1)!\,(n-k)!}$$

$$\begin{aligned}
&= \frac{(n-1)!}{(k-1)!\,(n-1-k)!} \cdot \left(\frac{1}{k} + \frac{1}{n-k}\right) \\
&= \frac{(n-1)!}{(k-1)!\,(n-1-k)!} \cdot \frac{n}{k(n-k)} \\
&= \frac{n!}{k!\,(n-k)!} \\
&= \binom{n}{k}
\end{aligned}$$

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**(b)** For any nonnegative integers $n$ and $m$,

$$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}.$$

*Proof.* For any fixed integer $m$, we will do induction on $n$.

1. **Base case:** $n = 0$.
   If $m = 0$, then

   $$\sum_{k=0}^{n} \binom{k}{m} = \sum_{k=0}^{0} \binom{k}{0} = \binom{0}{0} = 1 \quad \text{and} \quad \binom{n+1}{m+1} = \binom{1}{1} = 1.$$

   If $m > 0$, then

   $$\sum_{k=0}^{n} \binom{k}{m} = \binom{0}{m} = 0 \quad \text{and} \quad \binom{n+1}{m+1} = \binom{1}{m+1} = 0$$

   by definition, since $0 < m$ and $1 < m+1$.
2. **Inductive Step:** Show that for any integer $n \geq 0$, if the claim is true for $n$ then the claim must also be true for $n+1$.
   If the claim is true for $n$, then $\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$.
   We then have

   $$\sum_{k=0}^{n+1} \binom{k}{m} = \sum_{k=0}^{n} \binom{k}{m} + \binom{n+1}{m} = \binom{n+1}{m+1} + \binom{n+1}{m} = \binom{n+2}{m+1}.$$

   Here the second follows from the inductive hypothesis and the third equation follows from the result of part **(a)**.

Therefore, by induction, $\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$ for any nonnegative integers $n$ and $m$.

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**Problem 1-4.** Let $\theta$ be an angle for which $\sin\theta$ and $\cos\theta$ are rational. Prove that for all integers $n$, $\sin n\theta$ and $\cos n\theta$ are rational.

*Proof.*

First let us prove this for positive integers: $n \geq 1$. Clearly, we will use induction on $n$.

1. **Base case:** $n = 1$. Given.
2. **Inductive Step:** Assume the claim is true for $n$: $\sin n\theta$ and $\cos n\theta$ are rational. Prove the claim for $n+1$:

$$\sin(n+1)\theta = \sin(n\theta + \theta) = \sin n\theta \cos\theta + \cos n\theta \sin\theta$$

$$\cos(n+1)\theta = \cos(n\theta + \theta) = \cos n\theta \cos\theta - \sin n\theta \sin\theta$$

Since $\sin\theta, \cos\theta, \sin n\theta$ and $\cos n\theta$ are all rational, and the rationals are closed under addition and multiplication, it follows that $\sin(n+1)\theta$ and $\cos(n+1)\theta$ are rational.

This proves the theorem for positive integers.

Since $\sin(-n)\theta = -\sin n\theta$ and $\cos(-n)\theta = \cos n\theta$, $\sin n\theta$ and $\cos n\theta$ should be rational for all integers $n \leq -1$. For $n = 0$, $\sin 0 = 0$ and $\cos 0 = 1$ are clearly rational. Therefore,for all integers $n$, $\sin n\theta$ and $\cos n\theta$ are rational. ∎

**Problem 1-5.** Show that 5 divides $3^{2k} + 4^{k+1}$ for all integers $k \geq 0$.

*Proof.* By induction on $k$.

1. **Base case:** $k = 0$. We have $3^{2k} + 4^{k+1} = 1 + 4 = 5$, which is divisible by 5.
2. **Inductive Step:** Show that for any positive integer $k$, if 5 divides $3^{2k} + 4^{k+1}$, then 5 divides $3^{2(k+1)} + 4^{(k+1)+1}$.

$$3^{2(k+1)} + 4^{(k+1)+1} = 9 \cdot 3^{2k} + 4 \cdot 4^{k+1} = 4 \cdot (3^{2k} + 4^{k+1}) + 5 \cdot 3^{2k}.$$

If 5 divides $3^{2k} + 4^{k+1}$, then 5 divides $4 \cdot (3^{2k} + 4^{k+1}) + 5 \cdot 3^{2k}$.

We conclude that 5 divides $3^{2k} + 4^{k+1}$ for all integers $k \geq 0$. ∎