A Collection of Problems on Induction

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1 Prefatory Note

This note contains a collection of problems on induction. Each problem is ranked on a scale from 1 (least difficult) to 5 (most difficult). For the tougher problems a short hint is supplied. In addition, where possible the source of the problem is also referenced.

2 Problems

Problem 1 (3) *The powers-that-be at MIT decide they want Kilian Court tiled like a chessboard. A survey is done and it is found that Kilian Court is square:* $2^8 ft \times 2^8 ft$*. Further, the MIT management wishes to reserve 1 square foot of area, to be chosen at random, for the erection of a column in praise of the wisdom and glory of 6.042 TAs*[1]*. You are entrusted with this task and when the time comes to place an order for tiles you find that the only tiles available are L-shaped tiles consisting of three* $1ft \times 1ft$ *unit tiles fused together in an L-shape. Abstract this problem as follows: given a* $2^n \times 2^n$ *chessboard with one square removed from anywhere, is it possible to tile it using L-shaped tiles? Rise to the occasion and prove that this can be done thus paving the way for a successful transformation of Kilian Court [K93].*

Problem 2 (1) *Prove that any postage greater than or equal to eight cents can be made up exactly using only three and five cent stamps [T87].*

Problem 3 (5) *Generalizing problem 2, show that any integer greater than* $ab - a - b$ *can be represented as* $ax + by$*, where* a, b, x, y *are nonnegative integers [R93].*

Problem 4 (1) *Prove that there are* 2^n n*-digit binary sequences [T87].*

Problem 5 (1) *Generalize problem 4 to* m*-ary sequences and show that there are* m^n n*-digit* m*-ary sequences.*

[1] STEVE, I know you are reading this

Problem 6 (1) *Prove that there are* $n!$ *permutations of* $[1 \ldots n]$.

Problem 7 (2) *Prove that any set of* $n+1$ *numbers chosen from* $[1 \ldots 2n]$ *contains two numbers such that one divides the other.*

Problem 8 (2) *Let* n *straight lines divide the plane into regions. Prove that it is possible to paint the regions with the colours black and white so that no two adjoining regions are painted the same colour [T87].*

Problem 9 (2) *Given a collection of line segments on a line such that every two segments intersect prove that all the line segments intersect.*

Problem 10 (3) *Given a collection of convex sets in the plane such that every three sets have a nonempty intersection prove that all the sets have a nonempty intersection.*

Problem 11 (2) *Find and prove a formula for the sum of the angles of a convex polygon [MR91].*

Problem 12 (2) *Suppose* n *teams play a round-robin tournament in which there are no tie games. Then show that no matter what the individual outcomes, one can always number the teams* $t_1, t_2, \ldots, t_n$ *so that* t_1 *beat* t_2, t_2 *beat* t_3, *and so on through to* t_{n-1} *beat* t_n *[MR91].*

Problem 13 (2) *Show that for each integer* $n > 1$,

$$\underbrace{\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}_{n\,1\text{'s}}$$

is irrational [MR91].

Problem 14 (1) *Show that there is an infinity of primes [G89].*

Problem 15 (1) *Prove that* $\frac{n^7}{7} + \frac{n^3}{3} + \frac{11n}{21}$ *is an integer [G89].*

Problem 16 (2) *Prove that* $\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$ *[G89].*

Problem 17 (2) *Show that if a round robin tournament has an odd number of teams, it is possible for every team to win exactly half its games [MR91].*

Problem 18 (1) *What is the mistake in the following proof:*

We prove that in any collection of horses all horses are of the same colour.
Basis step. If we have a collection of only one horse then clearly all horses in the collection have the same colour.
Induction step. Given that any collection of k *horses is made up of identically*

coloured horses we wish to show that any collection of $k+1$ horses is also made up of identically coloured horses. Consider the $k+1$ horses standing in a line. By the induction hypothesis the first k of them have the same colour and also by the induction hypothesis the last k of them have the same colour, hence by the overlap and transitivity of equality all of them have the same colour [T87].

Problem 19 (2) *Find the flaw in the following proof:*
Proposition. $\forall x \neq 0, x^n = 1$.
Basis step. Since $x \neq 0$, $x^0 = 1$.
Induction step.
Induction hypothesis: $\forall 0 \leq m \leq k, x^m = 1$.
To show: $x^{k+1} = 1$.
By the induction hypothesis $x^k = x^{k-1} = 1$. *Hence*

$$x^{k+1} = \frac{x^k x^k}{x^{k-1}} = \frac{1 \times 1}{1} = 1$$

[T87].

Problem 20 (1) *Find the flaw in the following proof:*
Proposition. All natural numbers are equal.
We show by induction on n that if $\max(p,q) = n$, *then* $p = q$.
Basis step. If $\max(p,q) = 0$, *then we must have* $p = q = 0$.
Induction step.
Induction hypothesis: If $\max(p,q) = k$, *then* $p = q$.
To show: If $\max(p,q) = k+1$, *then* $p = q$. *Now* $\max(p-1, q-1) = k$ *so by the induction hypothesis* $p-1 = q-1$ *and hence* $p = q$ *[T87].*

Problem 21 (1) *Let the sequence* $f_1, f_2, \ldots$ *be defined by* $f_n = f_{n-1} + f_{n-2}$. *Prove that if f_1 and f_2 are integers then the sequence is an integer sequence.*

Problem 22 (4) *Let* $f_1 = f_2 = 1$ *in the sequence of problem 21. Prove that* $\forall$ *primes*$p \neq 5, p | f_{p-1}$ *or* $p | f_{p+1}$.

Problem 23 (1) *Let θ be an angle for which* $\sin\theta$ *and* $\cos\theta$ *are rational then prove that forall integers n* $\sin n\theta$ *and* $\cos n\theta$ *are rational MR91.*

Problem 24 (1) *There is a whole section of faulty induction proofs in [**?**].*

Problem 25 (4) *Consider n chords in a circle, no three concurrent. Prove that the interior of the circle is divided into* $\binom{n}{4} + \binom{n}{2} + \binom{n}{0}$.

Problem 26 (2) *Prove Euler's formula for plane graphs: #-vertices + #-faces = #-edges −1.*

Problem 27 (2) *Solve problem 25 using the result of problem 26.*

Problem 28 (5) *Give a linear-time median-finding algorithm.*

Problem 29 (2) *Prove that any spanning tree on n nodes has at most n edges.*

Problem 30 (2) *Prove that*

$$\frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$

.

Problem 31 (4) *Prove a lower bound of $n \log n$ on the size of merging networks.*

Problem 32 (5) *You are given a graph with light-bulbs and toggle switches at each node. Toggling the switch on a node toggles the status of the light bulbs at that node and all other neighbouring nodes. Initially all bulbs are turned off. Prove there is a setting of the switches which turns on all the bulbs.*

Problem 33 (2) *Prove Fermat's theorem: for all primes p, $a^p \cong a \bmod p$.*

References

[G89] R. P. Grimaldi. *Discrete and combinatorial mathematics: an applied introduction*. Addison Wesley Publishing Co., 1989.

[K93] M. Karchmer. Personal communication to Steve Ponzio, 1993.

[MR91] S. B. Maurer and A. Ralston. *Discrete algorithmic mathematics*. Addison Wesley Publishing Co., 1991.

[R93] R. Sundaram. Have to resolve these problems, 1993.

[T87] M. Townsend. *Discrete mathematics: applied combinatorics and graph theory*. Benjamin Cummings Publishing Co., 1987.