The Well Ordering Principle

Every *non-empty* set of *natural* numbers has a *minimum* element.

Do you believe this statement? Seems obvious, right? Well, it is. But don't fail to realize how tight it is. Crucially, it talks about a *non-empty* set —otherwise, it would clearly be false. And it also talks about *natural* numbers —otherwise, it might again be false: think for example what would happen with the integers, or even the positive rational numbers.

This statement has a name, it is called **the well-ordering principle**. And, as most things we give names to, it's important. Why? Because it is equivalent to induction.

Something can be proved by induction iff it can be proved by the well-ordering principle.

We could go on and give a general proof of this, but we won't. Instead, we'll just convince ourselves of it by going through an example. We'll reprove something that in the Notes (see Course Notes for Week 3, Theorem 4.1) was proved by induction. Read the next page.

For reference, here is the outline that a proof by the well-ordering principle has. (Compare it with the corresponding outline of a proof by strong induction given in Section 4.1 of the Notes.)

To prove that "$P(n)$ is true for all $n \in \mathbb{N}$" using the well-ordering principle:

- Use proof by contradiction.
- Assume that $P(n)$ has *counterexamples*. I.e., that $P(n)$ is false on at least one n.
- Define the set of counterexamples $C = \{n \in \mathbb{N} \mid \neg P(n)\}$.
- Invoke the well-ordering principle to select the minimum element c of C.
- Since c is *the smallest counterexample* to $P(n)$, conclude that both $\neg P(c)$ and $P(0), P(1), \ldots, P(c-1)$. Use these to arrive at a contradiction. Watch out: the list $0, 1, \ldots, c-1$ will contain no numbers at all if $c = 0$.
- Conclude that $P(n)$ must have no counterexamples. Namely, that $(\forall n)P(n)$.

Theorem. *For all* $n \in \mathbb{N}$*:* $1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

Proof. By contradiction. Assume that the theorem is *false*. Then, some natural numbers serve as *counterexamples* to it. Let's collect them in a set:

$$C = \left\{\, n \in \mathbb{N} \;\mid\; 1+2+3+\cdots+n \neq \tfrac{n(n+1)}{2} \,\right\}.$$

By our assumption that the theorem admits counterexamples, C is a non-empty set of natural numbers. So, by the well-ordering principle, C has a minimum element, call it c. That is, c is the *smallest counterexample* to the theorem.

Since c is a counterexample ($c \in C$), we know that

$$1+2+3+\cdots+c \neq \tfrac{c(c+1)}{2}.$$

Since c is the smallest counterexample (c minimum of C), we know the theorem holds for all natural numbers smaller than c. (Otherwise, at least one of them would also be in C and would therefore prevent c from being the minimum of C.) [$*$] In particular, the theorem is true for $c-1$. That is,

$$1+2+3+\cdots+(c-1) = \tfrac{(c-1)c}{2}.$$

But then, adding c to both sides we get

$$1+2+3+\cdots+(c-1)+c = \tfrac{(c-1)c}{2} + c = \tfrac{c^2-c+2c}{2} = \tfrac{c(c+1)}{2},$$

which means the theorem does hold for c, after all! That is, c is not a counterexample. But this is a contradiction. And we are done.

Well, almost. Our argument contains a bug. Everything we said after [$*$] bases on the fact that $c-1$ actually exists. That is, that there is indeed some natural number smaller than c. How do we know that? How do we know that c is not 0? Fortunately, this can be fixed. We know $c \neq 0$ because c is a counterexample whereas 0 is not, as $0 = 0(0+1)/2$. □