Quiz, Week 4

Problem 1. **(a)** What are the maxim*al* and minim*al* elements, if any, of the set, $\mathbb{N}$, of all nonnegative integers under divisibility? Is there a minim*um* or maxim*um* element?

(b) What are the minimal and maximal elements, if any, of the set of integers ≥ 2 under divisibility?

(c) Describe an infinite chain and an infinite antichain in the divisibility partial order on the nonnegative integers.

(d) Describe a partially ordered set (i.e., describe a set and then a partial order on this set) that has no minimal or maximal elements.

Problem 2. Consider two integers, a and b. We define the set, L, recursively, as follows:

- **Base case:** $0 \in L$.
- **Constructor cases:** if $x \in L$, then
  - $x + a \in L$,
  - $x - a \in L$,
  - $x + b \in L$,
  - $x - b \in L$.

Use structural induction to prove that all members of L are integer linear combinations of a and b. In other words, prove that

$$\text{for every } x \in L: \quad x = sa + tb \;\text{ for some integer coefficients } s \text{ and } t.$$

Appendix

Arithmetic

If m and n are nonnegative integers, we say m divides n if $km = n$ for some nonnegative integer k.

Relational Properties

A binary relation, R, on a set, A, is

- *transitive* if for every $a, b, c \in A$, aRb and bRc implies aRc.
- *asymmetric* if for every $a, b \in A$, aRb implies $\neg(bRa)$,
- *reflexive* if aRa for every $a \in A$,
- *antisymmetric* if for every $a \neq b \in A$, aRb implies $\neg(bRa)$,
- *irreflexive* if aRa holds for no $a \in A$.

Partial Orders

A binary relation is a *strict partial order* iff it is transitive and asymmetric. It is a *weak partial order* iff it is transitive, reflexive, and antisymmetric.

Let $\preceq$ be a weak (reflexive) partial order on a set, A.

- An element $a \in A$ is *minimal* iff there is no element in A that is $\preceq a$ except possibly a itself. Similarly, an element $a \in A$ is *maximal* iff there is no element in A that is $\succeq a$ except possibly a itself.
- An element $a \in A$ is a *lower bound* for a subset, $S \subseteq A$ iff $a \preceq s$ for every $s \in S$. Similarly, an element $a \in A$ is an *upper bound* for a subset, $S \subseteq A$ iff $s \preceq a$ for every $s \in S$.
- An element $a \in A$ is the *minimum* element iff a is a lower bound on A. Similarly, an element $a \in A$ is the *maximum* element iff a is an upper bound on A.
- Elements $a, b \in A$ are *comparable* iff either $a \preceq b$ or $b \preceq a$. Two elements are *incomparable* iff they are not comparable.
- A subset, $S \subseteq A$ is *totally ordered* iff every two distinct elements in S are comparable.
- A *chain* is a totally ordered subset of A.
- An *antichain* is a subset of A, such that no two elements in it are comparable.