Solutions to In-Class Problems Week 3, Tue.

Problem 1. A pair of 6.042 TAs, Grant and Sonya, have decided to devote some of their spare time this term to establishing dominion over the entire galaxy. Recognizing this as an ambitious project, they worked out the following table of tasks on the back of Sonya's copy of the lecture notes.

1. **Devise a logo** and cool imperial theme music - 8 days.
2. **Build a fleet** of Hyperwarp Stardestroyers out of eating paraphernalia swiped from Lobdell - 18 days.
3. **Seize control** of the United Nations - 9 days, after task #1.
4. **Get shots** for Grant's cat, Emilios - 11 days, after task #1.
5. **Open a Starbucks chain** for the army to get their caffeine - 10 days, after task #3
6. **Train an army** of elite interstellar warriors by dragging people to see *The Phantom Menace* dozens of times - 4 days, after tasks #3, #4, and #5.
7. **Launch the fleet** of Stardestroyers, crush all sentient alien species, and establish a Galactic Empire - 6 days, after tasks #2 and #6.
8. **Defeat Microsoft** - 8 days, after tasks #2 and #6.

We picture this information in Figure 1 below by drawing a point for each task, and labelling it with the name and weight of the task. An edge between two points indicates that the task for the higher point must be completed before beginning the task for the lower one.

(a) Give some valid order in which the tasks might be completed.

Solution. We can easily find several of them. The most natural one is valid, too: #1, #2, #3, #4, #5, #6, #7, #8. ■

Grant and Sonya want to complete all these tasks in the shortest possible time. However, they have agreed on some constraining work rules.

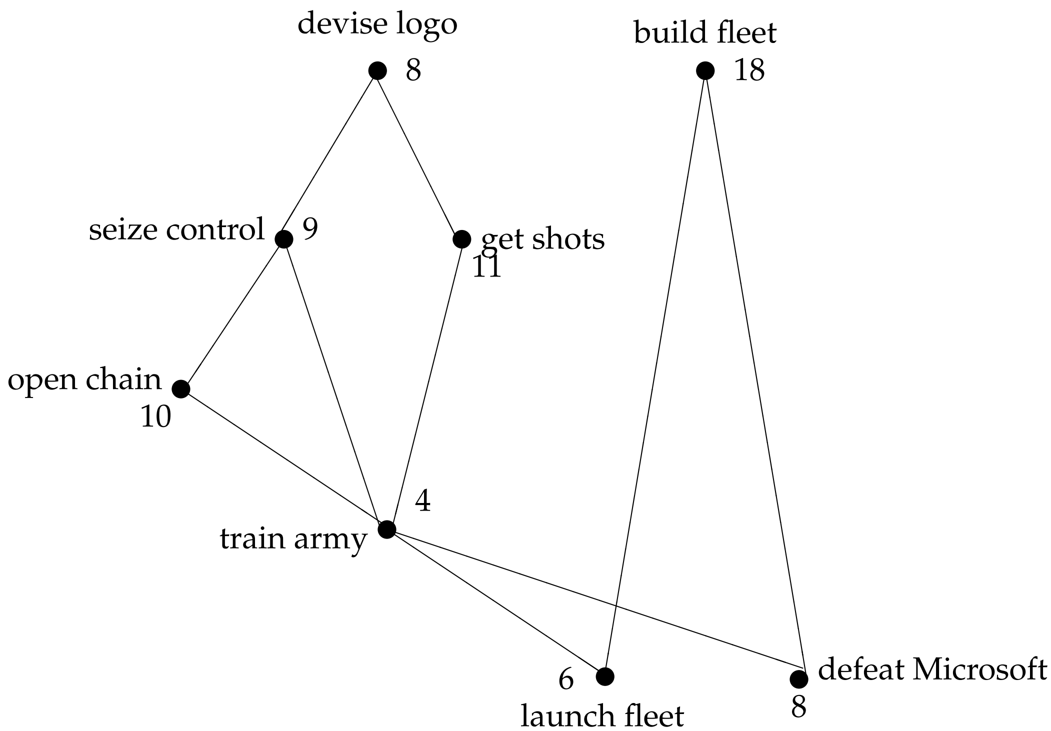

Figure 1: Graph representing the task precedence constraints.

- Only one person can be assigned to a particular task; they can not work together on a single task.
- Once a person is assigned to a task, that person must work exclusively on the assignment until it is completed. So, for example, Grant cannot work on building a fleet for a few days, run get shots for Emilios, and then return to building the fleet.

(b) Grant and Sonya want to know how long conquering the galaxy will take. Sonya suggests dividing the total number of days of work by the number of workers, which is two. What lower bound on the time to conquer the galaxy does this give, and why might the actual time required be greater?

Solution.

$$\frac{8+18+9+11+10+4+6+8}{2} = 37 \text{ days}$$

If working together and interrupting work on a task were permitted, then this answer would be correct. However, the rules may prevent Grant and Sonya from both working all the time. ■

(c) Grant proposes a different method for determining the duration of their project. He suggests looking at the duration of the "critical path", the most time-consuming sequence of tasks such that each depends on the one before. What lower bound does this give, and why might it also be too low?

Solution. The longest sequence of tasks is devising a logo (8 days), seizing the U. N. (9 days), opening a Starbucks (10 days), training the army (4 days), and then defeating Microsoft (8 days). Since these tasks must be done sequentially, galactic conquest will require at least 39 days.

If there were enough workers, this answer would be correct; however, with only two workers, Grant and Sonya may be unable to make progress on the critical path every day. ■

(d) What is the minimum number of days that Grant and Sonya need to conquer the galaxy? No proof is required.

Solution. 40 days. Tasks could be divided as follows:

Sonya: #1 (days 1-8), #3 (days 9-17), #4 (days 18-28), #8 (days 33-40).

Grant: #2 (days 1-18), #5 (days 19-28), #6 (days 29-32), #7 (days 33-38). ■

Problem 2. Verify that each of the following relations is a partial order. For each, indicate if it is strict, weak, and/or a total order. (Definitions are in the Appendix.)

(a) The superset relation, $\supseteq$, on some family of sets.

Solution. To prove $\supseteq$ is transitive, suppose $A \supseteq B$ and $B \supseteq C$. Then by definition of $\supseteq$, every $b \in B$ is also in A, and every $c \in C$ is also in B. Hence, every $c \in C$ is also in A, which proves that $A \supseteq C$. This proves that $\supseteq$ is transitive. It is also antisymmetric, because $A \supseteq B$ and $B \supseteq A$ implies $A = B$. Finally, it is reflexive, since $A \supseteq A$. So $\supseteq$ is a weak partial order.

There are lots of pairs of sets neither of which contains the other, so $\supseteq$ will not, in general, be a total order. ■

(b) The "divides" relation on natural numbers.

Solution. Suppose $k \mid m$ and $m \mid n$. So there are integers, k_1, k_2 such that $m = k_1 k$ and $n = k_2 m$. Hence, $n = k_2(k_1 k) = (k_2 k_1)k$, which means that $k \mid n$. So "divides" is transitive. Similarly, if $m \mid n$ and $n \mid m$, then $m = n$, so divides is antisymmetric, and so is a partial order. It is weak, because every integer divides itself. It is not total, since any two primes, for example, are incomparable. ■

Problem 3. **(a)** What are the maxim*al* and minim*al* elements, if any, of the set, $\mathbb{N}$, of all nonnegative integers under divisibility? Is there a minim*um* or maxim*um* element?

Solution. The minimum (and therefore unique minimal) element is 1 since 1 divides all natural numbers. The maximum (and therefore unique maximal) element is 0 since all numbers divide 0. ■

(b) What are the minimal and maximal elements, if any, of the set of integers ≥ 2 under divisibility?

Solution. All prime numbers are minimal elements, since no numbers divide them.

There is no maximal element, because for any $n \geq 2$, there is a "larger" number under the divisibility partial order, namely, mn, for any $m > 1$. ■

(c) What is the size of the longest chain that is guaranteed to exist in any partially ordered set of n elements? What about the largest antichain?

Solution. Chain size is 1 in the "discrete" partial order in which every two distinct elements are incomparable. Antichain size is 1 if the partial order is total. ■

Problem 4. Prove that a binary relation, R, on a set, A, is a strict partial order iff it is transitive and irreflexive.

Solution. If R is a strict p.o., then in particular, it is asymmetric. So aRa would imply $\neg(aRa)$, a contradiction. It follows that aRa cannot hold, so R is irreflexive.

Conversely, suppose R is transitive and irreflexive. We want to show that R is antisymmetric. So suppose it wasn't. Then there must be $a, b \in A$ such that aRb and bRa. Now transitivity implies that aRa, contradicting irreflexivity. This contradiction shows that R must be antisymmetric. ■

Appendix

Relational Properties

A binary relation, R, on a set, A, is

- *transitive* if for every $a, b, c \in A$, aRb and bRc implies aRc.
- *asymmetric* if for every $a, b \in A$, aRb implies $\neg(bRa)$,
- *reflexive* if aRa for every $a \in A$,
- *antisymmetric* if for every $a \neq b \in A$, aRb implies $\neg(bRa)$,
- *irreflexive* if aRa holds for no $a \in A$.

Partial Order

A binary relation is a *strict partial order* iff it is transitive and asymmetric. It is a *weak partial order* iff it is transitive, reflexive, and antisymmetric.

Let $\preceq$ be a weak (reflexive) partial order on a set, A.

- An element $a \in A$ is *minimal* iff there is no element in A that is $\preceq a$ except possibly a itself. Similarly, an element $a \in A$ is *maximal* iff there is no element in A that is $\succeq a$ except possibly a itself.
- An element $a \in A$ is a *lower bound* for a subset, $S \subseteq A$ iff $a \preceq s$ for every $s \in S$. Similarly, an element $a \in A$ is an *upper bound* for a subset, $S \subseteq A$ iff $s \preceq a$ for every $s \in S$.
- An element $a \in A$ is the *minimum* element iff a is a lower bound on A. Similarly, an element $a \in A$ is the *maximum* element iff a is an upper bound on A.
- Elements $a, b \in A$ are *comparable* iff either $a \preceq b$ or $b \preceq a$. Two elements are *incomparable* iff they are not comparable.
- A subset, $S \subseteq A$ is *totally ordered* iff every two distinct elements in S are comparable.
- A *chain* is a totally ordered subset of A.
- An *antichain* is a subset of A, such that no two elements in it are comparable.