In-Class Problems Week 2, Wed.

Problem 1. For each of the logical formulas, indicate whether or not it is true when the domain of discourse is $\mathbb{N}$ (the natural numbers 0, 1, 2, ...), $\mathbb{Z}$ (the integers), $\mathbb{Q}$ (the rationals), $\mathbb{R}$ (the real numbers), and $\mathbb{C}$ (the complex numbers).

$$\begin{array}{rrrcl} & \exists x & (x^2 & = & 2) \\ \forall x & \exists y & (x^2 & = & y) \\ \forall y & \exists x & (x^2 & = & y) \\ \forall x \neq 0 & \exists y & (xy & = & 1) \\ \exists x & \exists y & (x + 2y & = & 2) \wedge (2x + 4y = 5) \end{array}$$

Problem 2. The goal of this problem is to translate some assertions about binary strings into logic notation. The domain of discourse is the set of all finite-length binary strings: λ, 0, 1, 00, 01, 10, 11, 000, 001, ... (Here λ denotes the empty string.) In your translations, you may use all the ordinary logic symbols (including =), variables, and the binary symbols `0`, `1` denoting 0, 1.

A string like $\texttt{01}x\texttt{0}y$ of binary symbols and variables denotes the *concatenation* of the symbols and the binary strings represented by the variables. For example, if the value of x is `011` and the value of y is `1111`, then the value of $\texttt{01}x\texttt{0}y$ is the binary string `0101101111`.

Here are some examples of formulas and their English translations. Names for these predicates are listed in the third column so that you can reuse them in your solutions (as we do in the definition of the predicate NO-1S below).

Meaning	Formula	Name
x is a prefix of y	$\exists z\ (xz = y)$	PREFIX(x, y)
x is a substring of y	$\exists u \exists v\ (uxv = y)$	SUBSTRING(x, y)
x is empty or a string of 0's	$\neg$SUBSTRING$(\texttt{1}, x)$	NO-1S(x)

(a) x consists of three copies of some string.

(b) x is an even-length string of 0's.

(c) x does not contain both a 0 and a 1.

(d) x is the binary representation of $2^k + 1$ for some integer $k \geq 0$.

(e) An elegant, slightly trickier way to define NO-1S(x) is:

$$\text{PREFIX}(x, 0x). \quad (*)$$

Explain why (*) is true only when x is a string of 0's.

Problem 3. A media tycoon has an idea for an all-news television network called LNN: The Logic News Network. Each segment will begin with a definition of the domain of discourse and a few predicates. The day's happenings can then be communicated concisely in logic notation. For example, a broadcast might begin as follows:

> "THIS IS LNN. The domain of discourse is $\{\text{Bill}, \text{Monica}, \text{Ken}, \text{Linda}, \text{Betty}\}$. Let $D(x)$ be a predicate that is true if x is deceitful. Let $L(x, y)$ be a predicate that is true if x likes y. Let $G(x, y)$ be a predicate that is true if x gave gifts to y."

Complete the broadcast by translating the following statements into logic notation.

(a) If neither Monica nor Linda is deceitful, then Bill and Monica like each other.

(b) Everyone except for Ken likes Betty, and no one except Linda likes Ken.

(c) If Ken is not deceitful, then Bill gave gifts to Monica, and Monica gave gifts to someone.

(d) Everyone likes someone and dislikes someone else.

(e) How could you express "Everyone except for Ken likes Betty" using just propositional connectives *without* using any quantifiers ($\forall, \exists$)? Can you generalize to explain how *any* logical formula over this domain of discourse can be expressed without quantifiers? How big would the formula in the previous part be if it was expressed this way?

Problem 4. **(a)** Describe a counter-model demonstrating that

$$(\forall x \exists y.\ P(x, y)) \longrightarrow \forall z.\ P(z, z)$$

is not valid.

(b) Explain why

$$(\forall z.\ P(z, z)) \longrightarrow \forall x \exists y.\ P(x, y) \tag{1}$$

is valid.

Proof of a Validity about Quantifiers

Lemma. *The formula*

$$\forall x\ [P(x) \wedge Q(x)] \longrightarrow [(\forall y.\ P(y)) \wedge \forall z.\ Q(z)]$$

is valid.

Proof. Assume

$$\forall x\ [Q(x) \wedge P(x)] \tag{2}$$

holds (for some domain and interpretation of P and Q). We want to prove that

$$(\forall y.\ Q(y)) \wedge \forall z.\ P(z) \tag{3}$$

holds.

To do this, let c be an element of the domain. Then $Q(c) \wedge P(c)$ holds by hypothesis (4). In particular, $Q(c)$ holds. But since, c could have been any element of the domain, we conclude (by Universal Generalization) that

$$\forall y.\ Q(y) \tag{4}$$

holds. We conclude similarly that

$$\forall z.\ P(z) \tag{5}$$

holds. Now (6) and (7) immediately yield (5), as required. □