Problems for Recitation 8

The Chemist and The Maltster

At Guiness brewery in the early 1900's, W. S. Gosset (a chemist) and E. S. Beaven (a "maltster") were working to improve barley. Gosset and Beavan planned to grow several varieties of barley in a field and compare the yields. However, local variations in the fertility of the field might skew the results. Their solution was to divide the field into many small plots and grow each crop in several different places.

Similar thinking led to the use of *Latin squares* in experiment design. A Latin square is an $n \times n$ array of numbers such that each row and each column contains every number from 1 to n. For example, here is a 4×4 Latin square:

1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

You can imagine that this array is an agricultural field where each square is a small plot, and the number inside indicates the variety of barley planted there.

There are some nice connections between Latin squares and graph theory. For example, see if you can construct a graph G_n such that there is a one-to-one correspondence between $n \times n$ Latin squares and valid n-colorings of G_n.

(Recall that an ***n-coloring*** of a graph is a way of assigning one of n colors to each vertex so that vertices joined by an edge are assigned different colors.)

Perfect Matching from Hall's Theorem

The next question about Latin squares will require a corollary of Hall's Theorem. A bipartite graph is *regular* if every vertex on the left has degree c and every vertex on the right has degree d. Prove the following:

Corollary. *A regular bipartite graph has a matching for the vertices on the left if* $c \geq d > 0$.

Completing Latin Squares

Now we return to Latin squares.

Suppose your teammate wrote only three lines of a 5×5 Latin square before deciding to chuck it all and become a maltster:

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4

Can you fill in the last two lines to extend this "Latin rectangle" to a complete Latin square?

Show that filling in the next row of a Latin rectangle is equivalent to finding a matching in some bipartite graph.

Prove that a matching must exist in this bipartite graph and, consequently, a Latin rectangle can always be extended to a Latin square.