Notes for Recitation 7

1 RSA

In 1977, Ronald Rivest, Adi Shamir, and Leonard Adleman proposed a highly secure cryptosystem (called **RSA**) based on number theory. Despite decades of attack, no significant weakness has been found. (Well, none that you and me would know. . .) Moreover, RSA has a major advantage over traditional codes: the sender and receiver of an encrypted message need not meet beforehand to agree on a secret key. Rather, the receiver has both a ***secret key***, which she guards closely, and a ***public key***, which she distributes as widely as possible. To send her a message, one encrypts using her widely-distributed public key. Then she decrypts the message using her closely-held private key. The use of such a ***public key cryptography*** system allows you and Amazon, for example, to engage in a secure transaction without meeting up beforehand in a dark alley to exchange a key.

RSA Public-Key Encryption

Beforehand The receiver creates a public key and a secret key as follows.

1. Generate two distinct primes, p and q.
2. Let $n = pq$.
3. Select an integer e such that $\gcd(e, (p-1)(q-1)) = 1$.
   The *public key* is the pair (e, n). This should be distributed widely.
4. Compute d such that $de \equiv 1 \pmod{(p-1)(q-1)}$.
   The *secret key* is the pair (d, n). This should be kept hidden!

Encoding The sender encrypts message m to produce m' using the public key:

$$m' = m^e \text{ rem } n.$$

Decoding The receiver decrypts message m' back to message m using the secret key:

$$m = (m')^d \text{ rem } n.$$

2 Let's try it out!

You'll probably need extra paper. *Check your work carefully!*

- As a team, go through the **beforehand** steps.
  - Choose primes p and q to be relatively small, say in the range 10-20. In practice, p and q might contain several hundred digits, but small numbers are easier to handle with pencil and paper.
  - Try $e = 3, 5, 7, \ldots$ until you find something that works. Use Euclid's algorithm to compute the gcd.
  - Find d using the Pulverizer.

  When you're done, put your public key on the board. This lets another team send you a message.

- Now send an encrypted message to another team using their public key. Select your message m from the codebook below:

  2 = Greetings and salutations!

  3 = Yo, wassup?

  4 = You guys suck!

  5 = All your base are belong to us.

  6 = Someone on *our* team thinks someone on *your* team is kinda cute.

  7 = You *are* the weakest link. Goodbye.

- Decrypt the message sent to you and verify that you received what the other team sent!

- Explain how you could read messages encrypted with RSA if you could quickly factor large numbers.

  Solution. Suppose you see a public key (e, n). If you can factor n to obtain p and q, then you can compute d using the Pulverizer. This gives you the secret key (d, n), and so you can decode messages as well as the inteded recipient.

3 But does it really work?

A critical question is whether decrypting an encrypted message always gives back the original message! Mathematically, this amounts to asking whether:

$$m^{de} \equiv m \pmod{pq}.$$

Note that the procedure ensures that $de = 1 + k(p-1)(q-1)$ for some integer k.

- This congruence holds for all messages m. First, use Fermat's theorem to prove that $m \equiv m^{de} \pmod{p}$ for all m. (Fermat's Theorem says that $a^{p-1} \equiv 1 \pmod{p}$ if p is a prime that does not divide a.)

  Solution. If m is a multiple of p, then the claim holds because both sides are congruent to 0 mod p. Otherwise, suppose that m is not a multiple of p. Then:

  $$\begin{aligned} m^{1+k(p-1)(q-1)} &\equiv m \cdot (m^{p-1})^{k(q-1)} \pmod{p} \\ &\equiv m \cdot 1^{k(q-1)} \pmod{p} \\ &\equiv m \pmod{p} \end{aligned}$$

  The second step uses Fermat's theorem, which says that $m^{p-1} \equiv 1 \pmod{p}$ provided m is not a multiple of p.

- By the same argument, you can equally well show that $m \equiv m^{ed} \pmod{q}$. Show that these two facts together imply that $m \equiv m^{ed} \pmod{pq}$ for all m.

  Solution. We know that:

  $$\begin{aligned} p &\mid (m - m^{ed}), \\ q &\mid (m - m^{ed}). \end{aligned}$$

  Thus, both p and q appear in the prime factorization of $m - m^{ed}$. Therefore, $pq \mid (m - m^{ed})$, and so:

  $$m \equiv m^{ed} \pmod{pq}.$$