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Problems for Recitation 5

1 Problem: Well-ordering principle

Here is the geometric sum formula, which you proved in a previous recitation.

$$1 + r + r^2 + r^3 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Use the well-ordering principle to prove that, when $r \neq 1$, the formula is true for all $n \in \mathbb{N}$. *Prepare a complete, careful solution!*

2 Problem: A robot

A robot lives on a two dimensional grid and is free to walk around. However each move it makes is always one step north or south *and* one step east or west. Its purpose in life is to reach point $(1, 0)$. Unfortunately, the robot was born at point $(0, 0)$. Prove that it will never see how point $(1, 0)$ looks like.

3 Square Infection

The following problem is fairly tough until you hear a certain one-word clue. Then the solution is easy! Suppose that we have an $n \times n$ grid, where certain squares are *infected*. Here is an example where $n = 6$ and infected squares are marked $\times$.

$\times$				$\times$	
	$\times$				
		$\times$	$\times$		
		$\times$			
			$\times$		$\times$

Now the infection begins to spread in discrete time steps. Two squares are considered *adjacent* if they share an edge; thus, each square is adjacent to 2, 3 or 4 others. A square is infected in the next time step if either

- the square was previously infected, or
- the square is adjacent to *at least two* already-infected squares.

In the example, the infection spreads as shown below.

$\times$				$\times$	
	$\times$				
		$\times$	$\times$		
		$\times$			
			$\times$		$\times$

$\Rightarrow$

$\times$	$\times$			$\times$	
$\times$	$\times$	$\times$			
	$\times$	$\times$	$\times$		
		$\times$			
		$\times$	$\times$		
		$\times$	$\times$	$\times$	$\times$

$\Rightarrow$

$\times$	$\times$	$\times$		$\times$	
$\times$	$\times$	$\times$	$\times$		
$\times$	$\times$	$\times$	$\times$		
	$\times$	$\times$	$\times$		
		$\times$	$\times$	$\times$	
		$\times$	$\times$	$\times$	$\times$

Over the next few time-steps, the entire grid becomes infected.

Theorem. *An $n \times n$ grid can become completely infected only if at least n squares are initially infected.*

Prove this theorem using induction and some additional reasoning. If you are stuck, ask your recitation instructor for the one-word clue!