

## Notes for Recitation 4

### 1 Strong Induction

Recall the principle of strong induction:

**Principle of Strong Induction.** Let  $P(n)$  be a predicate. If

- $P(0)$  is true, and
- for all  $n \in \mathbb{N}$ ,  $P(0) \wedge P(1) \dots \wedge P(n)$  implies  $P(n + 1)$ ,

then  $P(n)$  is true for all  $n \in \mathbb{N}$ .

As an example, let's derive the fundamental theorem of arithmetic.

**Theorem 1.** *Every positive integer  $n \geq 2$  can be written as the product of primes.*

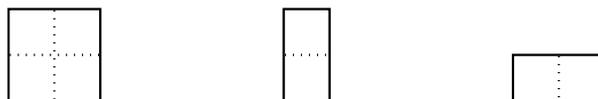
*Proof.* The proof is by strong induction. Let  $P(n)$  be the statement that  $n$  can be written as the product of primes.

*Base case:* 2 is itself a prime, so  $P(2)$  holds.

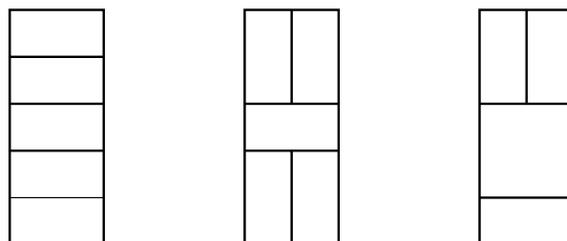
*Inductive case:* We show that  $P(2) \wedge P(3) \dots P(n) \Rightarrow P(n + 1)$  is true. Assume that  $P(2), P(3), \dots P(n)$  are true. Consider  $n + 1$ . If  $n + 1$  is a prime number,  $P(n + 1)$  holds. Otherwise,  $n + 1$  is a composite number; thus it has factors  $2 \leq u, v < n + 1$  such that  $u \cdot v = n + 1$ . By the inductive hypothesis,  $u = \prod_i p_i$  for primes  $p_i$  and  $v = \prod_j p_j$  for primes  $p_j$ , so therefore,  $n + 1 = \prod_i p_i \cdot \prod_j p_j$ , i.e.  $P(n + 1)$  is true. By the principle of strong induction,  $P(n)$  is true for all  $n \geq 2$ .  $\square$

## 2 Problem: Mini-Tetris

A *winning configuration* in the game of Mini-Tetris is a complete tiling of a  $2 \times n$  board using only the three shapes shown below:



For example, here are several possible winning configurations on a  $2 \times 5$  board:

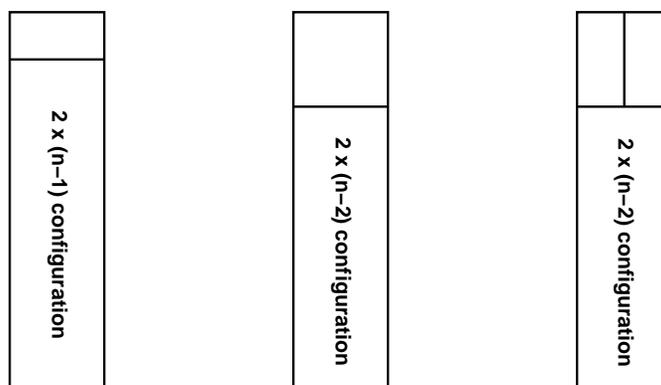


1. Let  $T_n$  denote the number of different winning configurations on a  $2 \times n$  board. Determine the values of  $T_1$ ,  $T_2$ , and  $T_3$ .

**Solution.**  $T_1 = 1$ ,  $T_2 = 3$ , and  $T_3 = 5$ .

2. Express  $T_n$  in terms of  $T_{n-1}$  and  $T_{n-2}$ .

**Solution.** Every winning configuration on a  $2 \times n$  board is of one of the following three types:



There are  $T_{n-1}$  winning configurations of the first type, and there are  $T_{n-2}$  winning configurations of the second and third types. Overall, the number of winning configurations on a  $2 \times n$  board is:

$$T_n = T_{n-1} + 2T_{n-2}$$

3. Using strong induction, prove that the number of winning configurations on a  $2 \times n$  Mini-Tetris board ( $n \geq 1$ ) is:

$$T_n = \frac{2^{n+1} + (-1)^n}{3}$$

**Solution.** The proof is by strong induction on  $n$ , with the induction hypothesis  $T_n = (2^{n+1} + (-1)^n)/3$ . The hypothesis is true for  $n = 1$  and  $n = 2$ , since we determined above that  $T_1 = 1 = (2^{1+1} + (-1)^1)/3$  and  $T_2 = 3 = (2^{2+1} + (-1)^2)/3$ .

Now suppose  $n \geq 2$  and assume that the hypothesis holds for all  $k < n$ . Starting from the equation derived in the preceding problem part, we have:

$$\begin{aligned} T_n &= T_{n-1} + 2T_{n-2} \\ &= \frac{2^n + (-1)^{n-1}}{3} + 2 \cdot \frac{2^{n-1} + (-1)^{n-2}}{3} \\ &= \frac{2^n - (-1)^n}{3} + \frac{2^n + 2(-1)^n}{3} \\ &= \frac{2^{n+1} + (-1)^n}{3} \end{aligned}$$

We use the induction hypothesis twice in the second step, and then simplify in the third and fourth steps. This completes the inductive step. By strong induction, the hypothesis holds for all  $n \geq 1$ .

### 3 Problem: Breaking a chocolate bar

We are given a chocolate bar with  $m \times n$  squares of chocolate, and our task is to divide it into  $mn$  individual squares. We are only allowed to split one piece of chocolate at a time using a vertical or a horizontal break.

For example, suppose that the chocolate bar is  $2 \times 2$ . The first split makes two pieces, both  $2 \times 1$ . Each of these pieces requires one more split to form single squares. This gives a total of three splits.

Use strong induction to conclude the following:

**Theorem.** *To divide up a chocolate bar with  $m \times n$  squares, we need at most  $mn - 1$  splits.*

**Solution.** This theorem does not immediately lend itself to an induction proof, since there are two variables. In general, propositions involving several natural-valued variables can often be proved by using a sort of nested induction. However, in this case, we can get by with a single-variable induction and a trick.

Intuitively, to break up a big chocolate bar, we need one split to make two pieces, and then we can break up the two pieces recursively. This suggests a proof using strong induction on the *size* of the chocolate bar, where size is measured in chocolate squares. Now instead of a problem involving two variables (the two dimensions), we have a problem in one variable (the size). With this simplification, we can prove the theorem using strong induction.

*Proof.* The proof is by strong induction on the size of the chocolate bar. Let  $P(k)$  be the proposition that a chocolate bar of size  $k$  requires at most  $k - 1$  splits.

*Base case,  $k = 1$ :*  $P(1)$  is true because there is only a single square of chocolate, and  $1 - 1 = 0$  splits are required.

*Induction step:* We suppose  $k \geq 1$  and any chocolate bar of size  $s$ , where  $1 \leq s \leq k$ , requires at most  $s - 1$  splits. We must now show there is a way to split a chocolate bar of size  $k + 1$  with at most  $k$  splits.

To do this, first break the chocolate bar of size  $k + 1$  into two smaller pieces of size  $p$  and  $q$  where  $p + q = k + 1$ . This is certainly possible because the size of the bar is at least two. Now the pieces of sizes  $p$  and  $q$  are between one and  $k$ , so by strong induction, breaking these two pieces into single squares requires only  $p - 1$  and  $q - 1$  splits, respectively. The total number of splits required to break the bar of size  $k + 1$  into single squares is therefore at most  $1 + (p - 1) + (q - 1) = p + q - 1 = (k + 1) - 1 = k$ .

This shows that  $P(k)$  implies  $P(k + 1)$ , and the claim is proved by strong induction.  $\square$

## 4 Problem: Fibonacci numbers

The Fibonacci numbers are defined as follows:

$$F_1 = 1, F_2 = 1, \text{ and for all } k \geq 3, F_k = F_{k-1} + F_{k-2}.$$

The first few terms of the Fibonacci sequence are:

$$1, 1, 2, 3, 5, 8, 13, 21, \dots$$

We can't find every single number in the Fibonacci sequence – for instance, 4 is not a number in the sequence. But can we express every  $n \geq 1$  as the sum of distinct terms in the Fibonacci sequence? Indeed, we can!

Use strong induction to prove the following:

**Theorem 2.** *Every  $n \geq 1$  can be expressed as the sum of distinct terms in the Fibonacci sequence.*

**Solution.**

*Proof.* We proceed by strong induction. Let  $P(n)$  be the statement that  $n$  can be written as the sum of distinct terms in the Fibonacci sequence.

**Base case:** 1 itself is a term in the Fibonacci sequence, so  $P(1)$  is true.

**Inductive case:** We want to show that  $P(1) \wedge P(2) \dots P(n) \Rightarrow P(n+1)$  is true. Assume that  $P(1), P(2) \dots P(n)$  all hold. Consider  $n+1$ . If  $n+1$  is a Fibonacci number, then we are done. If not, we have, for some  $k \geq 1$ ,

$$F_k < n+1 < F_{k+1}.$$

This gives us the equation:

$$n+1 = F_k + (n+1 - F_k)$$

. Now  $n+1 - F_k < n+1$ , so by the induction hypothesis,  $P(n+1 - F_k)$  is true, i.e.

$$n+1 - F_k = F_{j_1} + F_{j_2} + \dots$$

Furthermore, none of the  $F_{j_i}$  is  $F_k$ , since since  $n+1 - F_k < F_k$ . Therefore, we have that  $P(n+1)$  is true, i.e.:

$$n+1 = F_k + F_{j_1} + F_{j_2} + \dots$$

By the principle of strong induction,  $P(n)$  is true for all  $n$ .

□