Notes for Recitation 3

1 Induction

Recall the principle of induction:

Principle of Induction. Let $P(n)$ be a predicate. If

- $P(0)$ is true, and
- for all $n \in \mathbb{N}$, $P(n)$ implies $P(n+1)$,

then $P(n)$ is true for all $n \in \mathbb{N}$.

2 Problem: A Geometric Sum

Perhaps you encountered this classic formula in school:

$$1 + r + r^2 + r^3 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Use induction to prove that it is correct for all real values $r \neq 1$.

Prepare a complete, careful solution. You'll be passing your proof to another group for "constructive criticism"!'

Solution.

Proof. We use induction. Let $P(n)$ be the proposition that the following equation holds for all $r \neq 1$:

$$1 + r + r^2 + r^3 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Base case: $P(0)$ is true, because both sides of the equation are equal to 1.

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \in \mathbb{N}$. So assume that $P(n)$ is true, where n denotes an arbitrary natural number. We can reason as follows:

$$\begin{aligned} 1 + r + r^2 + r^3 + \ldots + r^n + r^{n+1} &= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} \\ &= \frac{1 - r^{n+1} + (1 - r) \cdot r^{n+1}}{1 - r} \\ &= \frac{1 - r^{n+2}}{1 - r} \end{aligned}$$

The first equation follows from the assumption $P(n)$, and the remaining steps are simplifications. This proves that $P(n+1)$ is also true. Therefore, $P(n)$ implies $P(n+1)$ for all $n \in \mathbb{N}$. By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

Note: You may have encountered a different proof of this formula. We'll write down a sequence of equations and then explain the reasoning.

$$\begin{aligned} S &= 1 + r + r^2 + r^3 + \ldots + r^n \\ rS &= r + r^2 + r^3 + \ldots + r^{n+1} \\ S - rS &= 1 - r^{n+1} \\ S &= \frac{1 - r^{n+1}}{1 - r} \end{aligned}$$

We define S on the first line, multiply by r to get the second equation, subtract the second equation from the first to get the third, and then solve for S. This gives the formula above!

This argument is great! It is a derivation of the formula rather than just a verification. But, at some level, we've only hidden the use of induction, since the operations we're doing on n-term sums are justified using— you guessed it— induction.

3 Problem: A False Proof

In lecture, we proved that:

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

But now we're going to prove a *contradictory* theorem!

False Theorem 1. *For all* $n \geq 0$,

$$2+3+4+\ldots+n=\frac{n(n+1)}{2}$$

Proof. We use induction. Let $P(n)$ be the proposition that $2+3+4+\ldots+n = n(n+1)/2$.

Base case: $P(0)$ is true, since both sides of the equation are equal to zero. (Recall that a sum with no terms is zero.)

Inductive step: Now we must show that $P(n)$ implies $P(n+1)$ for all $n \geq 0$. So suppose that $P(n)$ is true; that is, $2+3+4+\ldots+n = n(n+1)/2$. Then we can reason as follows:

$$\begin{aligned}
2+3+4+\ldots+n+(n+1) &= \left[2+3+4+\ldots+n\right]+(n+1) \\
&= \frac{n(n+1)}{2}+(n+1) \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

Above, we group some terms, use the assumption $P(n)$, and then simplify. This shows that $P(n)$ implies $P(n+1)$. By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

Where exactly is the error in this proof?

Solution. The short answer is that we failed to prove $P(0) \Rightarrow P(1)$, just as in the colored horses problem in lecture. In fact, once again, the error is rooted in the misleading nature of the "$\ldots$" notation.

More precisely, in the inductive step we are required to prove that $P(n)$ implies $P(n+1)$ for all $n \geq 0$. However, the argument given above breaks down when $n = 0$. Let's look more closely at the first equation in the indutive step to see why:

$$2+3+4+\ldots+n+(n+1) = \left[2+3+4+\ldots+n\right]+(n+1)$$

This seems completely innucuous; after all, we've only grouped terms! However, the left side contains *no terms* when $n = 0$. The "$\ldots$" is compeletely misleading in this case; 2, 3, 4, and $n+1$ are actually *not* in the sum. This misimpression becomes an error when we "pull out" the $(n+1)$ term on the right side, disregarding the fact that no such term actually existed on the left. Thus, for $n = 0$, the equation we've just written down says:

$$\underbrace{2+3+4+\ldots+n+(n+1)}_{=0} = \left[\underbrace{2+3+4+\ldots+n}_{=0}\right]+\underbrace{(n+1)}_{=1}$$

The assertion $0 = 0 + 1$ is false, and so we have not shown that $P(0)$ implies $P(1)$. There is no way to fix this problem and correctly prove that $P(0)$ implies $P(1)$, because actually $P(0)$ is true and $P(1)$ is false.

Thus, we've only established $P(0)$, $P(1) \Rightarrow P(2)$, $P(2) \Rightarrow P(3)$, and so forth. The induction argument falls apart because of the missing link $P(0) \not\Rightarrow P(1)$.

4 Problem: The Volcanic Island

There is a village on a volcanic island with $b \geq 1$ blue-eyed people and $g \geq 0$ green-eyed people. There are no mirrors and no one ever discusses eye color. Therefore, everyone knows the colors of everyone elses' eyes, but not their own. Good thing, because an islander who learns that he or she has blue eyes must leap into the volcano at the end of the same day!

The villagers live in happy ignorance for years. But one day an explorer arrives and loudly proclaims, "I see that at least one person here has blue eyes." Assuming that all the villagers are master logicians, what happens?

What happens is that at the end of the bth day, all the blue-eyed villagers jump into the volcano.

Use induction to prove that your conclusion is correct. We suggest the following hypothesis $P(n)$ that asserts all of the following are true on day n:

1. If $b > n$, then all blue-eyed people survive the day.
2. If $b = n$, then all blue-eyed people jump into the volcano.
3. If $b < n$, then all blue-eyed people are already dead.

Solution. Note that a green-eyed villager shouldn't ever conclude that she has blue eyes, since she doesn't, and we're assuming the villagers always reason correctly from what they know (and what they know from the explorer is also true). So no green-eyed person should ever jump into the volcano.

Theorem 2. *All the blue-eyed people jump into the volcano on day b.*

Proof. We use induction. Let $P(n)$ be the proposition that all of the following are true on day n:

1. If $b > n$, then all blue-eyed people survive the day.
2. If $b = n$, then all blue-eyed people jump into the volcano.
3. If $b < n$, then all blue-eyed people are already dead.

Base case: We must verify that the three parts of $P(n)$ hold on day $n = 1$.

1. Suppose $b > 1$. Consider events on day 1 from the perspective of a blue-eyed villager. The explorer says that someone has blue eyes, and she can indeed see at least one other person with blue eyes. Therefore, the facts available to her are consistent with her having either blue or green eyes. So she survives the day.
2. Suppose $b = 1$. The single blue-eyed villager sees no one else with blue eyes, concludes that he must have blue eyes, and jumps into the volcano. No one else jumps in because everyone else does see the blue-eyed villager and they have no reason at this point to think they too are blue-eyed.
3. This statement is vacuously true, because the if-part ($b < 1$) is false; the problem statement says that $b \geq 1$.

Therefore, $P(1)$ is true.

Inductive step: Now suppose that $P(n)$ is true where $n \geq 0$. We must verify the three parts of $P(n + 1)$.

1. Suppose $b > n + 1$. Then $b > n$ so all the blue-eyed people survived the preceding day by part 1. of $P(n)$. Furthermore, each blue-eyed villager can see at least $n+1 > n$ other blue-eyed people, so the observation that everyone survives is consistent with she herself having either blue or green eyes by $P(n)$ as well. Thus, each blue-eyed villager survives the day.

2. Suppose $b = n + 1$. Then $b > n$, so all the blue-eyed people survived the preceding day by part 1. of $P(n)$. Thus, on day $n + 1$ each blue-eyed villager knows $b > n$, but sees only n other people with blue eyes. Thus, each blue-eyed villager realizes that she has blue eyes and jumps into the volcano.

3. Suppose $b < n + 1$. Then either $b = n$ (in which case all blue-eyed people jumped into the volcano on day n by part 2. of $P(n)$) or else $b < n$ (in which case all blue-eyed-people were already dead on day n by part 3. of $P(n)$). In either case, all the blue-eyed people are already toast.

Therefore $P(n)$ implies $P(n + 1)$ for all $n \geq 0$.

By the principle of induction, $P(n)$ is true for all $n \geq 0$, and the theorem follows. $\square$