Problems for Recitation 3

1 Problem: A Geometric Sum

Perhaps you encountered this classic formula in school:

$$1 + r + r^2 + r^3 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Use induction to prove that it is correct for all real values $r \neq 1$.

Prepare a complete, careful solution. You'll be passing your proof to another group for "constructive criticism"!'

2 Problem: A False Proof

In lecture, we proved that:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

But now we're going to prove a *contradictory* theorem!

False Theorem 1. *For all* $n \geq 0$*,*

$$2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$$

Proof. We use induction. Let $P(n)$ be the proposition that $2 + 3 + 4 + \ldots + n = n(n+1)/2$.

Base case: $P(0)$ is true, since both sides of the equation are equal to zero. (Recall that a sum with no terms is zero.)

Inductive step: Now we must show that $P(n)$ implies $P(n+1)$ for all $n \geq 0$. So suppose that $P(n)$ is true; that is, $2 + 3 + 4 + \ldots + n = n(n+1)/2$. Then we can reason as follows:

$$\begin{aligned} 2 + 3 + 4 + \ldots + n + (n+1) &= \big[2 + 3 + 4 + \ldots + n\big] + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

Above, we group some terms, use the assumption $P(n)$, and then simplify. This shows that $P(n)$ implies $P(n+1)$. By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

Where exactly is the error in this proof?

3 Problem: The Volcanic Island

There is a village on a volcanic island with $b \geq 1$ blue-eyed people and $g \geq 0$ green-eyed people. There are no mirrors and no one ever discusses eye color. Therefore, everyone knows the colors of everyone elses' eyes, but not their own. Good thing, because an islander who learns that he or she has blue eyes must leap into the volcano at the end of the same day!

The villagers live in happy ignorance for years. But one day an explorer arrives and loudly proclaims, "I see that at least one person here has blue eyes." Assuming that all the villagers are master logicians, what happens?

What happens is that at the end of the bth day, all the blue-eyed villagers jump into the volcano.

Use induction to prove that your conclusion is correct. We suggest the following hypothesis $P(n)$ that asserts all of the following are true on day n:

1. If $b > n$, then all blue-eyed people survive the day.
2. If $b = n$, then all blue-eyed people jump into the volcano.
3. If $b < n$, then all blue-eyed people are already dead.