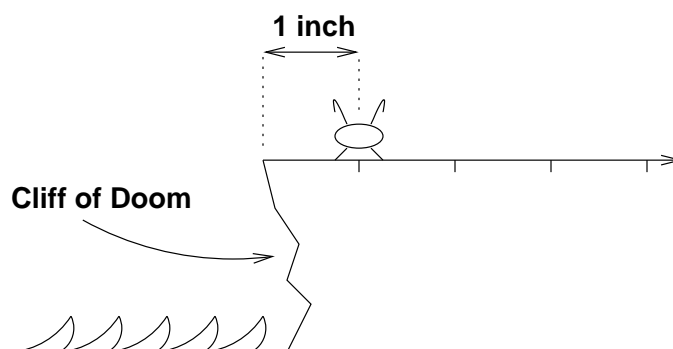


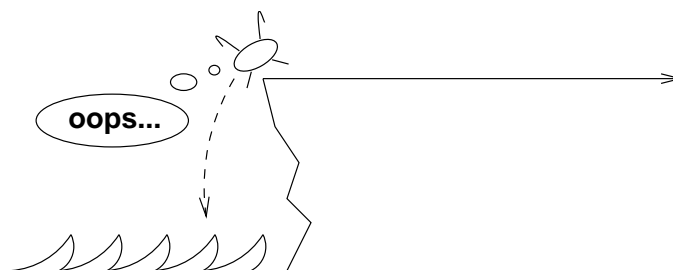
Notes for Recitation 23

1 Stencil the flea

There is a small flea named Stencil. To his right, there is an endless flat plateau. One inch to his left is the Cliff of Doom, which drops to a raging sea filled with flea-eating monsters.

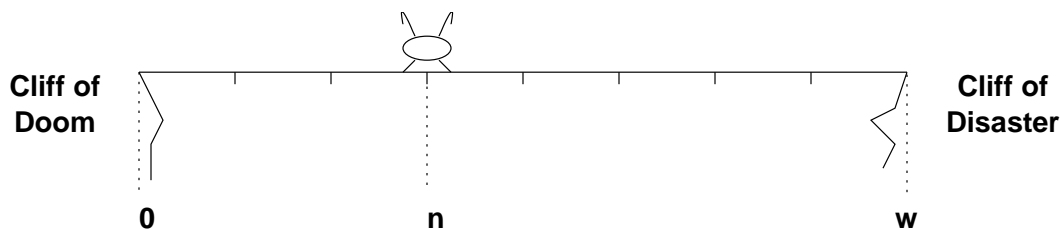


Each second, Stencil hops 1 inch to the right or 1 inch to the left with equal probability, independent of the direction of all previous hops. If he ever lands on the very edge of the cliff, then he teeters over and falls into the sea.



Our job is to analyze the life of Stencil. Does he have any chance of avoiding a fatal plunge? If not, how long will he hop around before he takes the plunge?

Problem 1. Let's begin with a simpler problem. Suppose that Stencil is n inches from the left side of an island w inches across:



In other words, Stencil starts at position n , for some $0 \leq n \leq w$ and there are cliffs at positions 0 and w . Let R_n be the probability he falls to the right off the Cliff of Disaster, given that he starts at position n .

- (a) What are the values of R_0 and R_n ? When $0 < n < w$, can you express R_n in terms of R_{n-1} and R_{n+1} ? (Hint: Total Probability!)

Solution. If $n = w$, he starts at position w and immediately falls from the Cliff of Disaster, so $R_w = 1$. On the other hand, if he starts at position 0, then he falls from the Cliff of Doom immediately, so $R_0 = 0$.

Now suppose that he stands somewhere in the middle of the island, so that $0 < n < w$. Then we can break the analysis of his fate into two cases based on the direction of his first hop:

- If his first hop is to the left, then he lands at position $n - 1$ and eventually falls off the Cliff of Disaster with probability R_{n-1} .
- On the other hand, if his first hop is to the right, then he lands at position $n + 1$ and eventually falls off the Cliff of Disaster with probability R_{n+1} .

Therefore, by the Total Probability Theorem, we have:

$$R_n = \frac{1}{2}R_{n-1} + \frac{1}{2}R_{n+1}$$

- (b) Solve the linear recurrence (you don't see any linear recurrence? talk to your TA!) to find R_n . (There is our usual guide on the last page.)

Solution. We rearrange the terms in the recurrence equation:

$$R_{n+1} = 2R_n - R_{n-1}$$

The characteristic equation is:

$$x^2 - 2x + 1 = 0$$

This equation has a double root at $x = 1$. There is no inhomogenous part, so the general solution has the form:

$$R_n = a \cdot 1^n + b \cdot n1^n = a + bn$$

Substituting in the boundary conditions $R_0 = 0$ and $R_w = 1$ gives two linear equations:

$$\begin{aligned} 0 &= a \\ 1 &= a + bw \end{aligned}$$

The solution to this system is $a = 0$, $b = 1/w$. Therefore, the solution to the recurrence is:

$$R_n = n/w$$

(c) So you know the probability that Stencil falls off the right side. Can you quickly deduce the probability that he will ... falls off the *left* side? ... lives on forever?

Solution. We exploit the symmetry of the problem: the probability that he falls off the *left* side starting at position n is the same as the probability that he falls of the *right* side starting at position $w - n$, which is $(w - n)/w$.

This is bad news. The probability that Stencil eventually falls off one cliff or the other is:

$$\frac{n}{w} + \frac{w - n}{w} = 1$$

There's no hope! The probability that he hops around on the island forever is zero.

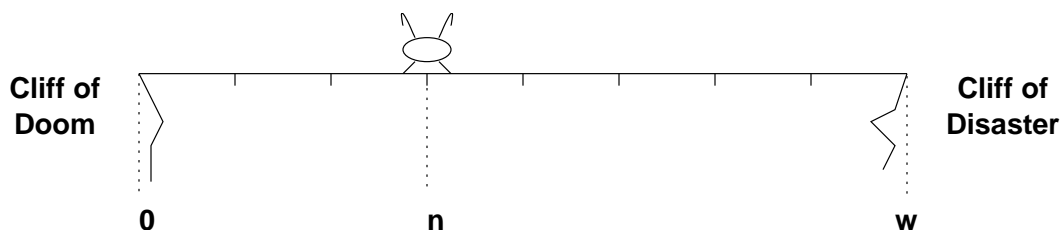
(d) Now let's go back to the original problem, where Stencil is 1 inch from the left edge of an infinite plateau. What is the probability that he lives on forever?

Solution. The probability that he eventually falls into the sea is:

$$\lim_{w \rightarrow \infty} \frac{w - 1}{w} = 1$$

Our little friend is doomed!

Problem 2. By now you must already know the tragic fate that awaits poor little Stencil the flea. On the bright side, though, Stencil may get to hop around for a while before he sinks beneath the waves. Let's find out how much he is expected to live. We should begin with the simpler setup as before:



Let X_n be the expected number of hops he takes before falling off a cliff.

- (a) What are the values of X_0 and X_w ? If $0 < n < w$, can you express X_n in terms of X_{n-1} and X_{n+1} ? (Hint: Total Expectation!)

Solution. If he starts at either edge of the island, then he dies immediately:

$$X_0 = 0$$

$$X_w = 0$$

If he starts somewhere in the middle of the island ($0 < n < w$), then we can again break down the analysis into two cases based on his first hop:

- If his first hop is to the left, then he lands at position $n - 1$ and can expect to live for another X_{n-1} steps.
- If his first hop is to the right, then he lands at position $n + 1$ and his expected lifespan is X_{n+1} .

Thus, by the Total Expectation Theorem and linearity, his expected lifespan is:

$$X_n = 1 + \frac{1}{2}X_{n-1} + \frac{1}{2}X_{n+1}$$

The leading 1 accounts for his first hop.

- (b) Now you should solve the recurrence.

Solution. We can rewrite the last line as:

$$X_{n+1} = 2X_n - X_{n-1} - 2$$

As before, the characteristic equation is:

$$x^2 - 2x + 1 = 0$$

There is a double-root at 1, so the homogenous solution has the form:

$$X_n = a + bn$$

There's an inhomogenous term, so we also need to find a particular solution. Since this term is a constant, we should try a particular solution of the form $X_n = c$ and then try $X_n = c + dn$ and then $X_n = c + dn + en^2$ and so forth. As it turns out, the first two possibilities don't work, but the third does. Substituting in this guess gives:

$$\begin{aligned} X_{n+1} &= 2X_n - X_{n-1} - 2 \\ c + d(n+1) + e(n+1)^2 &= 2(c + dn + en^2) - (c + d(n-1) + e(n-1)^2) - 2 \\ e &= -1 \end{aligned}$$

All the c and d terms cancel, so $X_n = c + dn - n^2$ is a particular solution for all c and d . For simplicity, let's take $c = d = 0$. Thus, our particular solution is $X_n = -n^2$.

Adding the homogenous and particular solutions gives the general form of the solution:

$$X_n = a + bn - n^2$$

Substituting in the boundary conditions $X_0 = 0$ and $X_w = 0$ gives two linear equations:

$$\begin{aligned} 0 &= a \\ 0 &= a + bw - w^2 \end{aligned}$$

The solution to this system is $a = 0$ and $b = w$. Therefore, the solution to the recurrence equation is:

$$X_n = wn - n^2 = n(w - n)$$

(c) Return to the original problem, where Stencil has the Cliff of Doom 1 inch to his left and an infinite plateau to his right: What is his expected lifespan there?

Solution. In this case, his expected lifespan is:

$$\lim_{w \rightarrow \infty} 1(w - 1) = \infty$$

Yes, Stencil is expected to live forever!

(d) Compare your answer to the previous part and your answer to the last part of the previous problem. Anything troublesome?

Solution. So, Stencil is certain to eventually fall off the cliff into the sea— but his expected lifespan is infinite! This sounds almost like a contradiction, but both answers are correct!

Here's an informal explanation. The probability that Stencil falls from the Cliff of Doom on the k -th step is approximately $1/k^{3/2}$. Thus, the probability that he falls eventually is:

$$\Pr(\text{falls off cliff}) \approx \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$$

You can verify by integration that this sum converges. The exact sum actually converges to 1. On the other hand, the expected time until he falls is:

$$\text{Ex}(\text{hops until fall}) \approx \sum_{k=1}^{\infty} k \cdot \frac{1}{k^{3/2}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

And you can verify by integration that this sum diverges. So our answers are compatible!