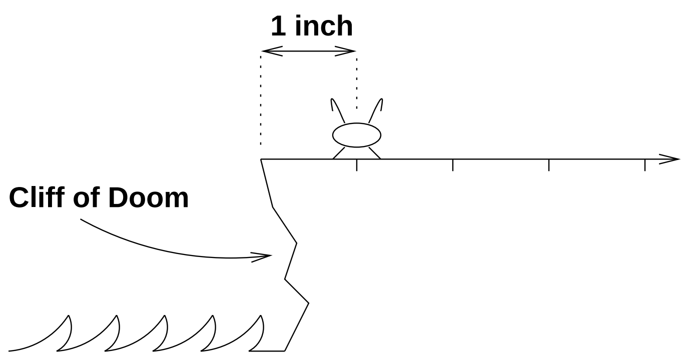
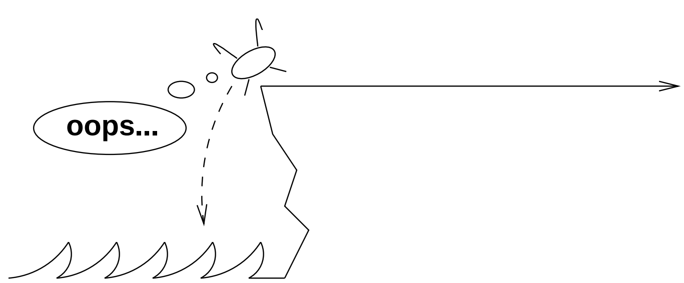
Problems for Recitation 23

1 Stencil the flea

There is a small flea named Stencil. To his right, there is an endless flat plateau. One inch to his left is the Cliff of Doom, which drops to a raging sea filled with flea-eating monsters.

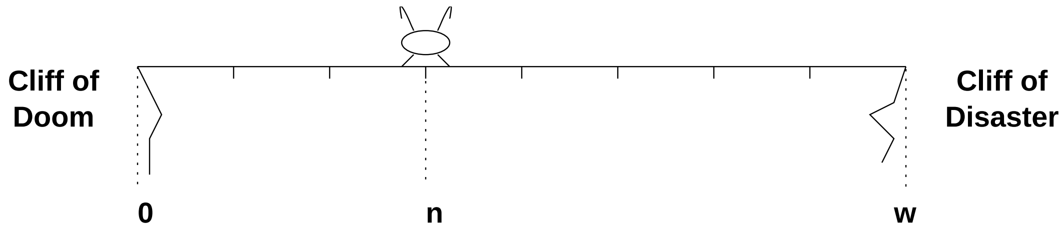


Each second, Stencil hops 1 inch to the right or 1 inch to the left with equal probability, independent of the direction of all previous hops. If he ever lands on the very edge of the cliff, then he teeters over and falls into the sea.



Our job is to analyze the life of Stencil. Does he have any chance of avoiding a fatal plunge? If not, how long will he hop around before he takes the plunge?

Problem 1. Let's begin with a simpler problem. Suppose that Stencil is n inches from the left side of an island w inches across:



In other words, Stencil starts at position n, for some $0 \leq n \leq w$ and there are cliffs at positions 0 and w. Let R_n be the probability he falls to the right off the Cliff of Disaster, given that he starts at position n.

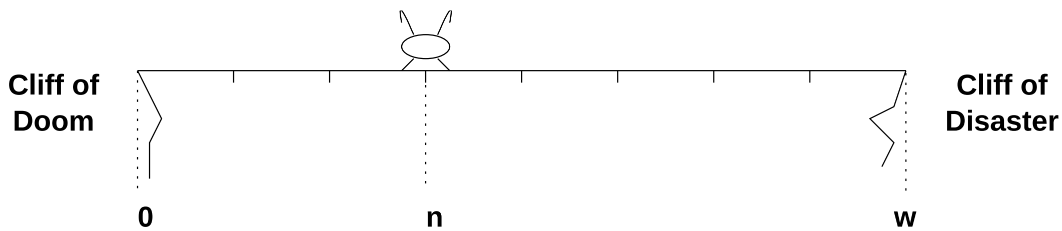
(a) What are the values of R_0 and R_n? When $0 < n < w$, can you express R_n in terms of R_{n-1} and R_{n+1}? (Hint: Total Probability!)

(b) Solve the linear recurrence (you don't see any linear recurrence? talk to your TA!) to find R_n. (There is our usual guide on the last page.)

(c) So you know the probability that Stencil falls off the right side. Can you quickly deduce the probability that he will . . . falls off the *left* side? . . . lives on forever?

(d) Now let's go back to the original problem, where Stencil is 1 inch from the left edge of an infinite plateau. What is the probability that he lives on forever?

Problem 2. By now you must already know the tragic fate that awaits poor little Stencil the flea. On the bright side, though, Stencil may get to hop around for a while before he sinks beneath the waves. Let's find out how much he is expected to live. We should begin with the simpler setup as before:



Let X_n be the expected number of hops he takes before falling off a cliff.

(a) What are the values of X_0 and X_w? If $0 < n < w$, can you express X_n in terms of X_{n-1} and X_n? (Hint: Total Expectation!)

(b) Now you should solve the recurrence.

(c) Return to the original problem, where Stencil has the Cliff of Doom 1 inch to his left and an infinite plateau to this right: What is his expected lifespan there?

(d) Compare your answer to the previous part and your answer to the last part of the previous problem. Anything troublesome?

Short Guide to Solving Linear Recurrences

A *linear recurrence* is an equation

$$\underbrace{f(n) = a_1 f(n-1) + a_2 f(n-2) + \ldots + a_d f(n-d)}_{\text{homogeneous part}} \qquad \underbrace{+g(n)}_{\text{inhomogeneous part}}$$

together with boundary conditions such as $f(0) = b_0$, $f(1) = b_1$, etc.

1. Find the roots of the *characteristic equation*:

   $$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_k$$

2. Write down the *homogeneous solution*. Each root generates one term and the homogeneous solution is the sum of these terms. A nonrepeated root r generates the term $c_r r^n$, where c_r is a constant to be determined later. A root r with multiplicity k generates the terms:

   $$c_{r_1} r^n, \quad c_{r_2} n r^n, \quad c_{r_3} n^2 r^n, \quad \ldots, \quad c_{r_k} n^{k-1} r^n$$

   where $c_{r_1}, \ldots, c_{r_k}$ are constants to be determined later.

3. Find a *particular solution*. This is a solution to the full recurrence that need not be consistent with the boundary conditions. Use guess and verify. If $g(n)$ is a polynomial, try a polynomial of the same degree, then a polynomial of degree one higher, then two higher, etc. For example, if $g(n) = n$, then try $f(n) = bn + c$ and then $f(n) = an^2 + bn + c$. If $g(n)$ is an exponential, such as 3^n, then first guess that $f(n) = c3^n$. Failing that, try $f(n) = bn3^n + c3^n$ and then $an^2 3^n + bn3^n + c3^n$, etc.

4. Form the *general solution*, which is the sum of the homogeneous solution and the particular solution. Here is a typical general solution:

   $$f(n) = \underbrace{c2^n + d(-1)^n}_{\text{homogeneous solution}} + \underbrace{3n + 1}_{\text{particular solution}}$$

5. Substitute the boundary conditions into the general solution. Each boundary condition gives a linear equation in the unknown constants. For example, substituting $f(1) = 2$ into the general solution above gives:

   $$\begin{aligned} 2 &= c \cdot 2^1 + d \cdot (-1)^1 + 3 \cdot 1 + 1 \\ \Rightarrow \quad -2 &= 2c - d \end{aligned}$$

   Determine the values of these constants by solving the resulting system of linear equations.