Problems for Recitation 19

Problem 1. Suppose that you flip three fair, mutually independent coins. Define the following events:

- Let A be the event that *the first* coin is heads.
- Let B be the event that *the second* coin is heads.
- Let C be the event that *the third* coin is heads.
- Let D be the event that *an even number of* coins are heads.

(a) Are these events pairwise independent?

(b) Are these events three-way independent? That is, does

$$\Pr(X \cap Y \cap Z) = \Pr(X) \cdot \Pr(Y) \cdot \Pr(Z)$$

always hold when X, Y, and Z are distinct events drawn from the set $\{A, B, C, D\}$?

(c) Are these events mutually independent?

Fact from lecture. If there are N days in a year and m people in a room, then the probability that no two people in the room have the same birthday is about:

$$e^{-m^2/(2N)}$$

Problem 2. Suppose that we create a a national database of DNA profiles. Let's make some simplistic assumptions:

- Each person can be classified into one of <u>20 billion</u> different "DNA types". (For example, you might be type #13,646,572,661 and the person next to you might be type #2,785,466,098.) Let $T(x)$ denote the type of person x.
- Each DNA type is equally probable.
- The DNA types of Americans are mutually independent.

(a) A congressman argues that there are only about 250 million Americans, so even if a profile for every American were stored in the database, the probability of even one coincidental match would be very small. How many profiles must the database actually contain in order for the probability of at least one coincidental match be about $1/2$?

(b) Person x is arrested for a crime that was committed by person y. At trial, jurors must determine whether $x = y$. The crime lab says x and y have the same DNA type. The prosecutor argues that the probability that x and y are different people is only 1 in 20 billion. Write the prosecutor's assertion in mathematical notation and explain her error.

Problem 3. Suppose there are 100 people in a room. Assume that their birthdays are independent and uniformly distributed. As stated in lecture notes, with probability $> 99\%$ there will be two that have the same birthday.

Now suppose you find out the birthdays of all the people in the room except one—call her "Jane"—and find all 99 dates to be different.

(a) What's wrong with the following argument:

> With probability greater than 99%, some pair of people in the room have the same birthday. Since the 99 people we asked all had different birthdays, it follows that with probability greater than 99% Jane has the same birthday as some other person in the room.

(b) What is the actual probability that Jane has the same birthday as some other person in the room?

Problem 4. There were n Immortal Warriors born into our world, but in the end *there can be only one*. The Immortals' original plan was to stalk the world for centuries, dueling one another with ancient swords in dramatic landscapes until only one survivor remained. However, after a thought-provoking discussion of probabilistic independence, they opt to give the following protocol a try:

1. The Immortals forge a coin that comes up heads with probability p.
2. Each Immortal flips the coin once.
3. If *exactly one* Immortal flips heads, then he or she is declared The One. Otherwise, the protocol is declared a failure, and they all go back to hacking each other up with swords.

(a) One of the Immortals (the Kurgan from the Russian steppe) argues that as n grows large, the probability that this protocol succeeds must tend to zero. Another (McLeod from the Scottish highlands) argues that this need not be the case, provided p is chosen *very carefully*. What does your intuition tell you?

(b) What is the probability that the experiment succeeds as a function of p and n?

(c) How should p, the bias of the coin, be chosen in order to maximize the probability that the experiment succeeds? (You're going to have to compute a derivative!)

(d) What is the probability of success if p is chosen in this way? What quantity does this approach when n, the number of Immortal Warriors, grows large?