Problems for Recitation 18

The *Law of Total Probability* is a handy tool for breaking down the computation of a probability into distinct cases. More precisely, suppose we are interested in the probability of an event E: $\Pr(E)$. Suppose also that the random experiment can evolve in two different ways; that is, two different cases X and $\overline{X}$ are possible. Suppose also that

- it is easy to find the probability *of each case*: $\Pr(X)$ and $\Pr\left(\overline{X}\right)$,
- it easy to find the probability of the event *in each case*: $\Pr(E \mid X)$ and $\Pr\left(E \mid \overline{X}\right)$.

Then finding the probability of E is only two multiplications and an addition away.

Theorem 1 (Law of Total Probability). *Let E and X be events, and $0 < \Pr(X) < 1$. Then*

$$\Pr(E) \quad = \quad \Pr(X) \cdot \Pr(E \mid X) \quad + \quad \Pr\left(\overline{X}\right) \cdot \Pr\left(E \mid \overline{X}\right)$$

Proof. Let's simplify the right-hand side.

$$\begin{aligned}
&\Pr(E \mid X) \cdot \Pr(X) + \Pr\left(E \mid \overline{X}\right) \cdot \Pr\left(\overline{X}\right) \\
&\qquad = \frac{\Pr(E \cap X)}{\Pr(X)} \cdot \Pr(X) + \frac{\Pr\left(E \cap \overline{X}\right)}{\Pr\left(\overline{X}\right)} \cdot \Pr\left(\overline{X}\right) \\
&\qquad = \Pr(E \cap X) + \Pr\left(E \cap \overline{X}\right) \\
&\qquad = \Pr(E)
\end{aligned}$$

The first step uses the definition of conditional probability. On the next-to-last line, we're adding the probabilities of all outcomes in E *and* X to the probabilities of all outcomes in E and *not* in X. Since every outcome in E is either in X or not in X, this is the sum of the probabilities of all outcomes in E, which equals $\Pr(E)$. $\square$

What happens if the experiment can evolve in more than two different ways? That is, what if there are n cases, $X_1, \ldots, X_n$, which are *mutually exclusive* (no two cases can happen simultaneously) and *collectively exhaustive* (at least one case must happen)? If it is still easy to find the probability *of each case* and the probability of the event *in each case*, then again finding $\Pr(E)$ is trivial.

Theorem 2. *Let E be an event and let $X_1, \ldots, X_n$ be disjoint events whose union exhausts the sample space. Then*

$$\Pr(E) = \sum_{i=1}^{n} \Pr(E \mid X_i) \cdot \Pr(X_i)$$

provided that $\Pr(X_i) \neq 0$.

Problem 1. There is a rare and deadly disease called *Nerditosis* which afflicts about 1 person in 1000. On symption is a compulsion to refer to everything— fields of study, classes, buildings, etc.— using numbers. It's horrible. As victims enter their final, downward spiral, they're awarded a degree from MIT. Two doctors claim that they can diagnose Nerditosis.

(a) Doctor X received his degree from Harvard Medical School. He practices at Massachusetts General Hospital and has access to the latest scanners, lab tests, and research. Suppose you ask Doctor X whether you have the disease.

- If you have Nerditosis, he says "yes" with probability 0.99.
- If you don't have it, he says "no" with probability 0.97.

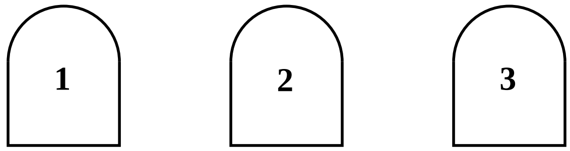
Let D be the event that you have the disease, and let E be the event that the diagnosis is erroneous. Use the Total Probability Law to compute $\Pr(E)$, the probability that Doctor X makes a mistake.

(b) "Doctor" Y received his genuine degree from a fully-accredited university for $49.95 via a special internet offer. He knows that Nerditosis stikes 1 person in 1000, but is a little shakey on how to interpret this. So if you ask him whether you have the disease, he'll helpfully say "yes" with probability 1 in 1000 regardless of whether you actually do or not.

Let D be the event that you have the disease, and let F be the event that the diagnosis is faulty. Use the Total Probability Law to compute $\Pr(F)$, the probability that Doctor Y made a mistake.

(c) Which doctor is more reliable?

Problem 2. A Barglesnort makes its lair in one of three caves:



The Barglesnort inhabits cave 1 with probability $\frac{1}{2}$, cave 2 with probability $\frac{1}{4}$, and cave 3 with probability $\frac{1}{4}$. A rabbit subsequently moves into one of the two unoccupied caves, selected with equal probability. With probability $\frac{1}{3}$, the rabbit leaves tracks at the entrance to its cave. (Barglesnorts are much too clever to leave tracks.) What is the probability that the Barglesnort lives in cave 3, given that there are no tracks in front of cave 2?

Use a tree diagram and the four-step method.

Problem 3. There is a deck of cards on the table. Either John or Mary shuffled it and we have no reason to believe in one case more than the other. Now, John is a well-known cheater with well-known preferences: he always steals the ace of diamonds while shuffling. Mary, on the other hand, is a very honest girl: a deck suffled by her is always a full 52-card deck.

(a) You pick the topmost card on the deck and you see a queen of hearts. Before you do any calculations: Who is more likely to have shuffled the deck? Explain.

(b) Now calculate. What is the probability that John has shuffled the deck? What is the probability that it has been Mary?

Like John, Peter is also a well-known cheater: when he shuffles the deck, he also steals a card from it; but (unlike John) he steals a *random* card. That is, every card is equally likely to be stolen when Peter is shuffling.

(c) Suppose you know that Mary shuffled the deck and you are about to pick the topmost card. What is the probability that you will see an ace?

(d) Suppose you know that Peter shuffled the deck and you are about to pick the topmost card. What is the probability that you will see an ace? (Hint: What is this probability if Peter steals an ace? What if Peter steals a non-ace?)

(e) Anything strange with the answers to parts (c) and (d)?