

Notes for Recitation 16

Problem 1. Find closed-form generating functions for the following sequences. Do not concern yourself with issues of convergence.

(a) $\langle 2, 3, 5, 0, 0, 0, 0, \dots \rangle$

Solution.

$$2 + 3x + 5x^2$$

(b) $\langle 1, 1, 1, 1, 1, 1, 1, \dots \rangle$

Solution.

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1 - x}$$

(c) $\langle 1, 2, 4, 8, 16, 32, 64, \dots \rangle$

Solution.

$$\begin{aligned} 1 + 2x + 4x^2 + 8x^3 + \dots &= (2x)^0 + (2x)^1 + (2x)^2 + (2x)^3 + \dots \\ &= \frac{1}{1 - 2x} \end{aligned}$$

(d) $\langle 1, 0, 1, 0, 1, 0, 1, 0, \dots \rangle$

Solution.

$$1 + x^2 + x^4 + x^6 + \dots = \frac{1}{1 - x^2}$$

(e) $\langle 0, 0, 0, 1, 1, 1, 1, 1, \dots \rangle$

Solution.

$$x^3 + x^4 + x^5 + x^6 + \dots = x^3(1 + x + x^2 + x^3 + \dots) = \frac{x^3}{1 - x}$$

(f) $\langle 1, 3, 5, 7, 9, 11, \dots \rangle$

Solution.

$$\begin{aligned}1 + x + x^2 + x^3 + \dots &= \frac{1}{1 - x} \\ \frac{d}{dx} 1 + x + x^2 + x^3 + \dots &= \frac{d}{dx} \frac{1}{1 - x} \\ 1 + 2x + 3x^2 + 4x^3 + \dots &= \frac{1}{(1 - x)^2} \\ 2 + 4x + 6x^2 + 8x^3 + \dots &= \frac{2}{(1 - x)^2} \\ 1 + 3x + 5x^2 + 7x^3 + \dots &= \frac{2}{(1 - x)^2} - \frac{1}{1 - x} \\ &= \frac{1 + x}{(1 - x)^2}\end{aligned}$$

Problem 2. Find a closed-form generating function for the sequence

$$(t_0, t_1, t_2, \dots)$$

where t_n is the number of different ways to compose a bag of n donuts subject to the following restrictions.

(a) All the donuts are chocolate and there are at least 3.

Solution.

$$\frac{x^3}{1-x}$$

(b) All the donuts are glazed and there are at most 4.

Solution.

$$\frac{1-x^5}{1-x}$$

(c) All the donuts are coconut and there are exactly 2.

Solution.

$$x^2$$

(d) All the donuts are plain and the number is a multiple of 4.

Solution.

$$\frac{1}{1-x^4}$$

(e) The donuts must be chocolate, glazed, coconut, or plain and:

- There must be at least 3 are chocolate donuts.
- There must be at most 4 glazed.
- There must be exactly 2 coconut.
- There must be a multiple of 4 plain.

Solution.

$$\frac{x^3}{1-x} \frac{1-x^5}{1-x} x^2 \frac{1}{1-x^4} = \frac{x^5(1+x^2+x^3+x^4)}{(1-x)(1-x^4)}$$

Problem 3. [20 points] A previous problem set introduced us to the Catalan numbers: $C_0, C_1, C_2, \dots$, where the n -th of them equals the number of balanced strings that can be built with $2n$ parentheses. Here is a list of the first several of them:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
C_n	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012

Then, in lecture we were all amazed to see that the decimal expansion of the irrational number $500000 - 1000\sqrt{249999}$

1.000001000002000005000014000042000132000429001430004862016796058786208012...

“encodes” these numbers! Now, there are many reasons why one may want to turn to religion, but this revelation is probably not a good one. Let’s explain why.

- (a) Let p_n be the number of balanced strings containing n (’s. Explain why the following recurrence holds:

$$p_0 = 1, \quad (\text{the empty string})$$

$$p_n = \sum_{k=1}^n p_{k-1} \cdot p_{n-k}, \quad \text{for } n \geq 1.$$

Solution. Note that every nonempty balanced string consists of a sequence of one or more balanced strings. The first balanced string in the sequence must begin with a (and end with a “matching”). That is, any balanced string, r_n , with $n \geq 1$ (’s consists of a balanced string, s_{k-1} , enclosed in brackets and containing $k-1$ (’s, followed by a balanced string, t_{n-k} , with $n-k$ (’s:

$$r_n = (s_{k-1}) \text{ followed by } t_{n-k}$$

where $1 \leq k \leq n$. This observation leads directly to the recurrence.

- (b) Now consider the generating function for the number of balanced strings:

$$P(x) = p_0 + p_1x + p_2x^2 + p_3x^3 + \dots$$

Prove that

$$P(x) = xP(x)^2 + 1.$$

Solution. We can verify this equation using the recurrence relation.

$$\begin{aligned}
 xP(x)^2 + 1 &= x(p_0 + p_1x + p_2x^2 + p_3x^3 + \dots)^2 + 1 \\
 &= x(p_0^2 + (p_0p_1 + p_1p_0)x + (p_0p_2 + p_1p_1 + p_2p_0)x^2 + \dots) + 1 \\
 &= 1 + p_0^2x + (p_0p_1 + p_1p_0)x^2 + (p_0p_2 + p_1p_1 + p_2p_0)x^3 + \dots \\
 &= 1 + p_1x + p_2x^2 + p_3x^3 + \dots \\
 &= P(x)
 \end{aligned}$$

(c) Find a closed-form expression for the generating function $P(x)$.

Solution. Given that $P(x) = xP(x)^2 + 1$, the quadratic formula implies that

$$P(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

If x is small, then $P(x)$ should be about $p_0 = 1$. Therefore, the correct choice of sign is

$$P(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

(d) Show that $P(1/1000000) = 500000 - 1000\sqrt{249999}$.

Solution.

$$\begin{aligned} P(1/1000000) &= \frac{1 - \sqrt{1 - 4/1000000}}{2/1000000} \\ &= 500000 - 500000\sqrt{\frac{249999}{250000}} \\ &= 500000 - 1000\sqrt{249999} \end{aligned}$$

(e) Explain why the digits of this irrational number encode these successive numbers of balanced strings.

Solution. Suppose that we symbolically carry out the substitution done in the preceding problem part.

$$\begin{aligned} P(x) &= p_0 + p_1x + p_2x^2 + p_3x^3 + \dots \\ P(10^{-6}) &= p_0 + p_110^{-6} + p_210^{-12} + p_310^{-18} + \dots \end{aligned}$$

Thus, p_0 appears in the units position, p_1 appears in the millionths position, p_2 appears in the trillionths position, and so forth.

Problem 4. Consider the following recurrence equation:

$$T_n = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 2T_{n-1} + 3T_{n-2} & (n \geq 2) \end{cases}$$

Let $f(x)$ be a generating function for the sequence $\langle T_0, T_1, T_2, T_3, \dots \rangle$.

(a) Give a generating function in terms of $f(x)$ for the sequence:

$$\langle 1, \quad 2, \quad 2T_1 + 3T_0, \quad 2T_2 + 3T_1, \quad 2T_3 + 3T_2, \dots \rangle$$

Solution. We can break this down into a linear combination of three sequences:

$$\begin{aligned} \langle 1, \quad 2, \quad 0, \quad 0, \quad 0, \quad \dots \rangle &= 1 + 2x \\ \langle 0, \quad T_0, \quad T_1, \quad T_2, \quad T_3, \quad \dots \rangle &= xf(x) \\ \langle 0, \quad 0, \quad T_0, \quad T_1, \quad T_2, \quad \dots \rangle &= x^2f(x) \end{aligned}$$

In particular, the sequence we want is very nearly generated by $1 + 2x + 2xf(x) + 3x^2f(x)$. However, the second term is not quite correct; we're generating $2 + 2T_0 = 4$ instead of the correct value, which is 2. We correct this by subtracting $2x$ from the generating function, which leaves:

$$1 + 2xf(x) + 3x^2f(x)$$

(b) Form an equation in $f(x)$ and solve to obtain a closed-form generating function for $f(x)$.

Solution. The equation

$$f(x) = 1 + 2xf(x) + 3x^2f(x)$$

equates the left sides of all the equations defining the sequence $T_0, T_1, T_2, \dots$ with all the right sides. Solving for $f(x)$ gives the closed-form generating function:

$$f(x) = \frac{1}{1 - 2x - 3x^2}$$

(c) Expand the closed form for $f(x)$ using partial fractions.

Solution. We can write:

$$1 - 2x - 3x^2 = (1 + x)(1 - 3x)$$

Thus, there exist constants A and B such that:

$$f(x) = \frac{1}{1 - 2x - 3x^2} = \frac{A}{1 + x} + \frac{B}{1 - 3x}$$

Now substituting $x = 0$ and $x = 1$ gives the system of equations:

$$\begin{aligned} 1 &= A + B \\ -\frac{1}{4} &= \frac{A}{2} - \frac{B}{2} \end{aligned}$$

Solving the system, we find that $A = 1/4$ and $B = 3/4$. Therefore, we have:

$$f(x) = \frac{1/4}{1+x} + \frac{3/4}{1-3x}$$

(d) Find a closed-form expression for T_n from the partial fractions expansion.

Solution. Using the formula for the sum of an infinite geometric series gives:

$$f(x) = \frac{1}{4} (1 - x + x^2 - x^3 + x^4 - \dots) + \frac{3}{4} (1 + 3x + 3^2x^2 + 3^3x^3 + 3^4x^4 + \dots)$$

Thus, the coefficient of x^n is:

$$T_n = \frac{1}{4} \cdot (-1)^n + \frac{3}{4} \cdot 3^n$$