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Problems for Recitation 16

Problem 1. Find closed-form generating functions for the following sequences. Do not concern yourself with issues of convergence.

(a) $\langle 2, 3, 5, 0, 0, 0, 0, \ldots \rangle$

(b) $\langle 1, 1, 1, 1, 1, 1, 1, \ldots \rangle$

(c) $\langle 1, 2, 4, 8, 16, 32, 64, \ldots \rangle$

(d) $\langle 1, 0, 1, 0, 1, 0, 1, 0, \ldots \rangle$

(e) $\langle 0, 0, 0, 1, 1, 1, 1, 1, \ldots \rangle$

(f) $\langle 1, 3, 5, 7, 9, 11, \ldots \rangle$

Problem 2. Find a closed-form generating function for the sequence

$$(t_0, t_1, t_2, \ldots)$$

where t_n is the number of different ways to compose a bag of n donuts subject to the following restrictions.

(a) All the donuts are chocolate and there are at least 3.

(b) All the donuts are glazed and there are at most 4.

(c) All the donuts are coconut and there are exactly 2.

(d) All the donuts are plain and the number is a multiple of 4.

(e) The donuts must be chocolate, glazed, coconut, or plan and:

- There must be at least 3 are chocolate donuts.
- There must be at most 4 glazed.
- There must be exactly 2 coconut.
- There must be a multiple of 4 plain.

Problem 3. [20 points] A previous problem set introduced us to the Catalan numbers: $C_0, C_1, C_2, \ldots$, where the n-th of them equals the number of balanced strings that can be built with $2n$ paretheses. Here is a list of the first several of them:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
C_n	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012

Then, in lecture we were all amazed to see that the decimal expansion of the irrational number $500000 - 1000\sqrt{249999}$

$$1.000001000002000005000014000042000132000429001430004862016796058786208012\ldots$$

"encodes" these numbers! Now, there are many reasons why one may want to turn to religion, but this revelation is probably not a good one. Let's explain why.

(a) Let p_n be the number of balanced strings containing n **(**'s. Explain why the following recurrence holds:

$$p_0 = 1, \qquad \text{(the empty string)}$$

$$p_n = \sum_{k=1}^{n} p_{k-1} \cdot p_{n-k}, \qquad \text{for } n \geq 1.$$

(b) Now consider the generating function for the number of balanced strings:

$$P(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots.$$

Prove that

$$P(x) = xP(x)^2 + 1.$$

(c) Find a closed-form expression for the generating function $P(x)$.

(d) Show that $P(1/1000000) = 500000 - 1000\sqrt{249999}$.

(e) Explain why the digits of this irrational number encode these successive numbers of balanced strings.

Problem 4. Consider the following recurrence equation:

$$T_n = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 2T_{n-1} + 3T_{n-2} & (n \geq 2) \end{cases}$$

Let $f(x)$ be a generating function for the sequence $\langle T_0, T_1, T_2, T_3, \ldots \rangle$.

(a) Give a generating function in terms of $f(x)$ for the sequence:

$$\langle 1, \quad 2, \quad 2T_1 + 3T_0, \quad 2T_2 + 3T_1, \quad 2T_3 + 3T_2, \ldots \rangle$$

(b) Form an equation in $f(x)$ and solve to obtain a closed-form generating function for $f(x)$.

(c) Expand the closed form for $f(x)$ using partial fractions.

(d) Find a closed-form expression for T_n from the partial fractions expansion.