Problems for Recitation 15

Problem 1. Learning to count takes practice!

(a) In how many different ways can Blockbuster arrange 64 copies of *13 conversations about one thing*, 96 copies of *L'Auberge Espagnole* and 1 copy of *Matrix Revolutions* on a shelf? What if they are to be arranged in 5 shelves?

(b) Find the number of 5-card hands with exactly three aces.

(c) Find the number of 5-card hands in which every suit appears at most twice.

(d) In how many ways can 20 indistinguishable pre-frosh be stored in four different crates if each crate must contain an *even* number of pre-frosh?

(e) How many paths are there from point $(0, 0)$ to $(50, 50)$ if every step increments one coordinate and leaves the other unchanged and there are impassable boulders sitting at points $(10, 10)$ and $(20, 20)$?

(f) In how many ways can the 72 students in 6.042 be divided into 18 groups of 4?

(g) Set A has r elements and set B has n elements. How many functions are there from A to B? How many of them are injective (one-to-one)? How many of them are bijective?

Problem 2. A pizza house is having a promotional sale. Their commercial reads:

> We offer 9 different toppings for your pizza! Buy 3 large pizzas at the regular price, and you can order each one with any combination of toppings absolutely free. That's $22,369,621$ different ways to design your order!

The ad writer was a former Harvard student who had evaluated the formula $(2^9)^3/3!$ on her calculator and gotten close to $22,369,621$. Unfortunately, $(2^9)^3/3!$ is obviously not an integer, so clearly something is wrong. What? In particular, did she overcount or did she undercount? What is the correct number?

Combinatorial proofs of identities

Recall the basic plan for a combinatorial proof of an identity $x = y$:

1. Define a set S.
2. Show that $|S| = x$ by counting one way.
3. Show that $|S| = y$ by counting another way.
4. Conclude that $x = y$.

Problem 3. You want to choose a team of m people from a pool of n people for your startup company, and from these m people you want to choose k to be the team managers. You took 6.042, so you know you can do this in

$$\binom{n}{m}\binom{m}{k}$$

ways. But your CFO, who went to Harvard Business School, comes up with the formula

$$\binom{n}{k}\binom{n-k}{m-k}.$$

Before doing the reasonable thing —dump on your CFO or Harvard Business School— you decide to check his answer against yours.

(a) Start by giving an *algebraic proof* that your CFO's formula agrees with yours.

(b) Now give a *combinatorial argument* proving this same fact.

Problem 4. Now try the following, more interesting theorem:

$$n2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}$$

(a) Start with a combinatorial argument. *Hint*: let S be the set of all sequences in $\{0, 1, *\}^n$ containing exactly one $*$.

(b) How would you prove it algebraically?