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Problems for Recitation 14

Counting Rules

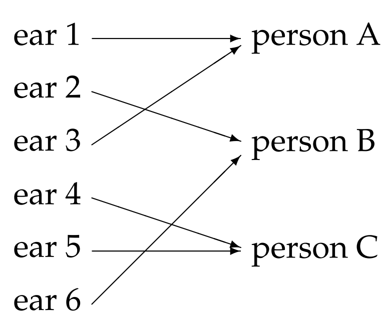
Rule 1 (Generalized Product Rule). *Let S be a set of length-k sequences. If there are:*

- n_1 *possible first entries,*
- n_2 *possible second entries for each first entry,*
- n_3 *possible third entries for each combination of first and second entries, etc.*

then:

$$|S| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$

A ***k-to-1 function*** maps exactly k elements of the domain to every element of the range. For example, the function mapping each ear to its owner is 2-to-1:



Rule 2 (Division Rule). *If $f : A \to B$ is k-to-1, then $|A| = k \cdot |B|$.*

The Generalized Product Rule

Problem 1. Solve the following counting problems using the generalized product rule.

(a) Next week, I'm going to get really fit! On day 1, I'll exercise for 5 minutes. On each subsequent day, I'll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 9, 11, 12. How many such sequences are possible?

(b) An r-***permutation*** of a set is a sequence of r distinct elements of that set. For example, here are all the 2-permutations of $\{a, b, c, d\}$:

$$\begin{array}{ccc} (a,b) & (a,c) & (a,d) \\ (b,a) & (b,c) & (b,d) \\ (c,a) & (c,b) & (c,d) \\ (d,a) & (d,b) & (d,c) \end{array}$$

How many r-permutations of an n-element set are there? Express your answer using factorial notation.

(c) How many $n \times n$ matrices are there with *distinct* entries drawn from $\{1, \ldots, p\}$, where $p \geq n^2$?

The Tao of BOOKKEEPPER

Problem 2. In this problem, we seek enlightenment through contemplation of the word $BOOKKEEPER$.

(a) In how many ways can you arrange the letters in the word $POKE$?

(b) In how many ways can you arrange the letters in the word BO_1O_2K? Observe that we have subscripted the O's to make them distinct symbols.

(c) Suppose we map arrangements of the letters in BO_1O_2K to arrangements of the letters in $BOOK$ by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

O_2BO_1K	
KO_2BO_1	$BOOK$
O_1BO_2K	$OBOK$
KO_1BO_2	$KOBO$
BO_1O_2K	$\ldots$
BO_2O_1K	
$\ldots$	

(d) What kind of mapping is this, young grasshopper?

(e) In light of the Division Rule, how many arrangements are there of $BOOK$?

(f) Very good, young master! How many arrangements are there of the letters in $KE_1E_2PE_3R$?

(g) Suppose we map each arrangement of $KE_1E_2PE_3R$ to an arrangement of $KEEPER$ by erasing subscripts. List all the different arrangements of $KE_1E_2PE_3R$ that are mapped to $REPEEK$ in this way.

(h) What kind of mapping is this?

(i) So how many arrangements are there of the letters in $KEEPER$?

(j) *Now you are ready to face the BOOKKEEPER!*
How many arrangements of $BO_1O_2K_1K_2E_1E_2PE_3R$ are there?

(k) How many arrangements of $BOOK_1K_2E_1E_2PE_3R$ are there?

(l) How many arrangements of $BOOKKE_1E_2PE_3R$ are there?

(m) How many arrangements of $BOOKKEEPER$ are there?

(n) How many arrangements of $VOODOODOLL$ are there?

(o) **(IMPORTANT)** How many n-bit sequences contain k zeros and $(n-k)$ ones?

This quantity is denoted $\binom{n}{k}$ and read "n choose k". You will see it almost every day in 6.042 from now until the end of the term.

Remember well what you have learned: subscripts on, subscripts off.
This is the Tao of Bookkeeper.

Problem 3. Solve the following counting problems. Define an appropriate mapping (bijective or k-to-1) between a set whose size you know and the set in question.

(a) (IMPORTANT) In how many ways can k elements be chosen from an n-element set $\{x_1, x_2, \ldots, x_n\}$?

(b) How many different ways are there to select a dozen donuts if four varieties are available?

(c) How many paths are there from $(0, 0)$ to $(10, 20)$ consisting of right-steps (which increment the first coordinate) and up-steps (which increment the second coordinate)?

(d) An independent living group is hosting eight pre-frosh, affectionately known at $P_1, \ldots, P_8$ by the permanent residents. Each pre-frosh must must be assigned a task: 2 must wash pots, 2 must clean the kitchen, 1 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. In how many ways can $P_1, \ldots, P_8$ be put to productive use?

(e) In how many ways can Mr. and Mrs. Grumperson distribute 13 indistinguishable pieces of coal to their two— no, three!— children for Christmas?

(f) How many solutions over the natural numbers are there to the equation:

$$x_1 + x_2 + \ldots + x_{10} \quad \leq \quad 100$$

(g) (Quiz 2, Fall '03) Suppose that two identical 52-card decks of are mixed together. In how many ways can the cards in this double-size deck be arranged?