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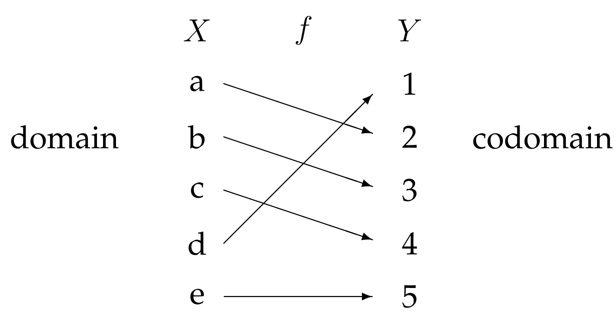
# Problems for Recitation 13

## Basic Counting Notions

A ***bijection*** or ***bijective function*** is a function $f : X \to Y$ such that every element of the codomain is related to exactly one element of the domain. Here is an example of a bijection:




**Rule 1 (Bijection Rule).** *If there exists a bijection $f : A \to B$, then $|A| = |B|$.*

**Rule 2 (Sum Rule).** *If $A_1, \ldots, A_n$ are disjoint sets, then:*

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} |A_k|$$

**Rule 3 (Product Rule).** *If $P_1, P_2, \ldots P_n$ are sets, then:*

$$|P_1 \times P_2 \times \cdots \times P_n| = |P_1| \cdot |P_2| \cdots |P_n|$$

**Rule 4 (Pigeonhole Principle).** *If $|X| > |Y|$, then for every function $f : X \to Y$ there exist two different elements of $X$ that are mapped to the same element of $Y$.*

> "If more than $n$ pigeons are assigned to $n$ holes, then there must exist two pigeons assigned to the same hole."

# Sum and Product Rules

**Problem 1.** A license plate consists of either:

- 3 letters followed by 3 digits (standard plate)
- 5 letters (vanity plate)

Let $L$ be the set of all possible license plates.

**(a)** Express $L$ in terms of

$$\mathcal{A} = \{A, B, C, \ldots, Z\}$$
$$\mathcal{D} = \{0, 1, 2, \ldots, 9\}$$

using unions ($\cup$) and set products ($\times$).

**(b)** Compute $|L|$, the number of different license plates, using the sum and product rules.

# Bijections

**Problem 2.** For each part below, describe a bijection between the two sets mentioned. The existence of such a bijection proves that the two sets are the same size.

A good approach is to describe an element of the first set using variables and then describe the corresponding element of the second set in terms of those variables. For example, we might describe a bijecton from ways of selecting a dozen doughnuts from five varieties to a 16-bit string with four 1's as follows:

> Map a dozen doughnuts consisting of:
>
> $c$ chocolate, $l$ lemon-filled, $s$ sugar, $g$ glazed, and $p$ plain
>
> to the sequence:
>
> $$\underbrace{0\ldots0}_{c}\quad 1\quad \underbrace{0\ldots0}_{l}\quad 1\quad \underbrace{0\ldots0}_{s}\quad 1\quad \underbrace{0\ldots0}_{g}\quad 1\quad \underbrace{0\ldots0}_{p}$$

*Everyone in your group should write out complete answers— you'll all benefit from the practice!*

**(a)** Describe a bijection between the set of $30$-bit sequences with 10 zeros and 20 ones and paths from $(0,0)$ to $(10,20)$ consisting of right-steps (which increment the first coordinate) and up-steps (which increment the second coordinate).

**(b)** Find a bijection between the set of $n$-bit sequences and the set of all subsets of $\{x_1, x_2, \ldots, x_n\}$.

**(c)** Mr. and Mrs. Grumperson have collected 13 identical pieces of coal as Christmas presents for their beloved children, Lucy and Spud. Describe a bijection between the set of all ways of distributing the 13 coal pieces to the two children and the set of 14-bit sequences with exactly 1 one.

**(d)** On Christmas Eve, Mr. and Mrs. Grumperson remember that they have a third child, little Bottlecap, locked in the attic. Describe a bijection between the set of all ways of distributing the 13 coal pieces to the three children and the set of 15-bit sequences with exactly 2 ones.

**(e)** On reflection, Mr. and Mrs. Grumperson decide that each of their three children should receive *at least two* pieces of coal for Christmas. Describe a bijection between the set of all ways of distributing the 13 coal pieces to the three Grumperson children given this constraint and the set of 9-bit sequences with exactly 2 ones.

**(f)** Describe a bijection between the set of 110-bit sequences with exactly 10 ones and solutions over the natural numbers to the equation:

$$x_1 + x_2 + \cdots + x_{10} \leq 100$$

**(g)** Describe a bijection between solutions to the inequality in the preceding problem part and sequences $(y_1, y_2, \ldots, y_{10})$ such that:

$$0 \leq y_1 \leq y_2 \leq \cdots \leq y_{10} \leq 100$$

# Pigeonhole Principle

**Problem 3.** Solve the following problems using the pigeonhole principle. For each problem, try to identify the *pigeons*, the *pigeonholes*, and a *rule* assigning each pigeon to a pigeonhole.

**(a)** In a room of 500 people, there exist two who share a birthday.

**(b)** Every MIT ID number starts with a 9 (we think). Suppose that each of the 75 students in 6.042 sums the nine digits of his or her ID number. Must two people arrive at the same sum?

**(c)** In every set of 100 integers, there exist two whose difference is a multiple of 37.

**(d)** For any five points in a unit square, there are two points at distance less than $\frac{1}{\sqrt{2}}$.

**(e)** For any five points in an equilateral triangle of side length 2, there are two points at distance less than $1$.

**(f)** Let $\{a_1, \ldots, a_{201}\}$ be a set of natural numbers less than 300. Then there are $i, j$ such that $\frac{a_i}{a_j} = 3^k$ for some $k > 0$.