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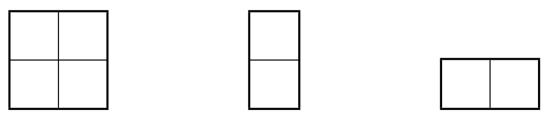
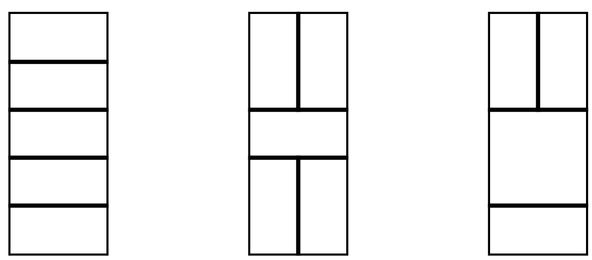
Problems for Recitation 12

1 Mini-Tetris (again)

Remember Mini-Tetris from Recitation 4? Here is an overview: A *winning configuration* in the game is a complete tiling of a $2 \times n$ board using only the three shapes shown below:



For example, the several possible winning configurations on a 2×5 board include:



In that past recitation, we had defined T_n to be the number of different winning configurations on a $2 \times n$ board. Then we had to inductively prove T_n equals some particular closed form expression. Remember that expression? Probably not. But no damage, now you can find it on your own.

(a) Determine the values of T_1, T_2, and T_3.

(b) Find a recurrence equation that expresses T_n in terms of T_{n-1} and T_{n-2}.

(c) Find a closed-form expression for T_n.

2 Inhomogeneous linear recurrences

Find a closed-form solution to the following linear recurrence.

$$
\begin{aligned}
T_0 &= 0 \\
T_1 &= 1 \\
T_n &= T_{n-1} + T_{n-2} + 1 \qquad (*)
\end{aligned}
$$

(a) First find the general solution to the corresponding homogenous recurrence.

(b) Now find a particular solution to the inhomogenous recurrence.

(c) The complete solution to the recurrence is the homogenous solution plus the particular solution. Use the initial conditions to find the coefficients.

(d) Therefore, the complete solution to the recurrence is:

3 Back to homogeneous ones

Let's get back to homogeneous linear recurrences. Find a closed-form solution to this one.

$$S_0 = 0$$
$$S_1 = 1$$
$$S_n = 6S_{n-1} - 9S_{n-2}$$

Anything strange?

Short Guide to Solving Linear Recurrences

A *linear recurrence* is an equation

$$\underbrace{f(n) = a_1 f(n-1) + a_2 f(n-2) + \ldots + a_d f(n-d)}_{\text{homogeneous part}} \qquad \underbrace{+g(n)}_{\text{inhomogeneous part}}$$

together with boundary conditions such as $f(0) = b_0$, $f(1) = b_1$, etc.

1. Find the roots of the *characteristic equation*:

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_k$$

2. Write down the *homogeneous solution*. Each root generates one term and the homogeneous solution is the sum of these terms. A nonrepeated root r generates the term $c_r r^n$, where c_r is a constant to be determined later. A root r with multiplicity k generates the terms:

$$c_{r_1} r^n, \quad c_{r_2} n r^n, \quad c_{r_3} n^2 r^n, \quad \ldots, \quad c_{r_k} n^{k-1} r^n$$

   where $c_{r_1}, \ldots, c_{r_k}$ are constants to be determined later.

3. Find a *particular solution*. This is a solution to the full recurrence that need not be consistent with the boundary conditions. Use guess and verify. If $g(n)$ is a polynomial, try a polynomial of the same degree, then a polynomial of degree one higher, then two higher, etc. For example, if $g(n) = n$, then try $f(n) = bn + c$ and then $f(n) = an^2 + bn + c$. If $g(n)$ is an exponential, such as 3^n, then first guess that $f(n) = c3^n$. Failing that, try $f(n) = bn3^n + c3^n$ and then $an^2 3^n + bn3^n + c3^n$, etc.

4. Form the *general solution*, which is the sum of the homogeneous solution and the particular solution. Here is a typical general solution:

$$f(n) = \underbrace{c2^n + d(-1)^n}_{\text{homogeneous solution}} + \underbrace{3n + 1}_{\text{particular solution}}$$

5. Substitute the boundary conditions into the general solution. Each boundary condition gives a linear equation in the unknown constants. For example, substituting $f(1) = 2$ into the general solution above gives:

$$\begin{aligned} 2 &= c \cdot 2^1 + d \cdot (-1)^1 + 3 \cdot 1 + 1 \\ \Rightarrow \quad -2 &= 2c - d \end{aligned}$$

   Determine the values of these constants by solving the resulting system of linear equations.