Problems for Recitation 11

1 The Quest

An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine d days walk from the nearest oasis. In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to create water caches out in the desert.

For example, if the shrine were $2/3$ of a day's walk into the desert, then she could recover the Holy Grail with the following strategy. She leaves the oasis with 1 gallon of water, travels $1/3$ day into the desert, caches $1/3$ gallon, and then walks back to the oasis— arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks $1/3$ day into the desert, tops off her water supply by taking the $1/3$ gallon in her cache, walks the remaining $1/3$ day to the shine, grabs the Holy Grail, and then walks for $2/3$ of a day back to the oasis— again arriving with no water to spare.

But what if the shrine were located farther away?

(a) What is the most distant point that the explorer can reach and return from if she takes only 1 gallon from the oasis.?

(b) What is the most distant point the explorer can reach and return form if she takes only 2 gallons from the oasis? No proof is required; just do the best you can.

(c) What about 3 gallons? (Hint: First, try to establish a cache of 2 gallons *plus* enough water for the walk home as far into the desert as possible. Then use this cache as a springboard for your solution to the previous part.)

(d) How can the explorer go as far as possible is she withdraws n gallons of water? Express your answer in terms of the Harmonic number H_n, defined by:

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$$

(e) Use the fact that

$$H_n \sim \ln n$$

to approximate your previous answer in terms of logarithms.

(f) Suppose that the shrine is $d = 10$ days walk into the desert. Relying on your approximate answer, how many days must the explorer travel to recover the Holy Grail?

2 Asymptotic notation

(a) Which of these symbols Θ O Ω o ω can go in these boxes?

$$2n + \log n = \boxed{}(n)$$

$$\log n = \boxed{}(n)$$

$$\sqrt{n} = \boxed{}(\log^{300} n)$$

$$n2^n = \boxed{}(n)$$

$$n^7 = \boxed{}(1.01^n)$$

(b) Indicate which of the following holds for each pair of functions $f(n), g(n)$ in the table below; $k \geq 1$, $\epsilon > 0$, and $c > 1$ are constants. Be prepared to justify your answers.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$	$f = \Theta(g)$	$f \sim g$
2^n	$2^{n/2}$						
$\sqrt{n}$	$n^{\sin n\pi/2}$						
$\log(n!)$	$\log(n^n)$						
n^k	c^n						
$\log^k n$	n^ϵ						

cheat sheet

Definitions. Intuitively and precisely the notations mean the following:

$f = \Theta(g)$	f grows *as fast* as g	There exists n_0 and $c_1, c_2 > 0$ such that for all $n > n_0$: $c_1 g(n) \le \lvert f(n) \rvert \le c_2 g(n)$.
$f = O(g)$	f grows *no faster* than g	There exists n_0 and $c > 0$ such that for all $n > n_0$: $\lvert f(n) \rvert \le c g(n)$.
$f = \Omega(g)$	f grows *no slower* than g	There exists n_0 and $c > 0$ such that for all $n > n_0$: $c g(n) \le \lvert f(n) \rvert$.
$f = o(g)$	f grows *slower* than g	For all $c > 0$, there exists n_0 such that for all $n > n_0$: $\lvert f(n) \rvert \le c g(n)$.
$f = \omega(g)$	f grows *faster* than g	For all $c > 0$, there exists n_0 such that for all $n > n_0$: $c g(n) \le \lvert f(n) \rvert$.
$f \sim g$	f/g approaches 1	$\lim_{n\to\infty} f(n)/g(n) = 1$

Relationships. Some asymptotic relationships between functions imply others:

$$f = O(g) \text{ and } f = \Omega(g) \Leftrightarrow f = \Theta(g) \qquad f = o(g) \Rightarrow f = O(g)$$
$$f = O(g) \Leftrightarrow g = \Omega(f) \qquad f = \omega(g) \Rightarrow f = \Omega(g)$$
$$f = o(g) \Leftrightarrow g = \omega(f) \qquad f \sim g \Rightarrow f = \Theta(g)$$

Limits. If the $\lim_{n\to\infty} f(n)/g(n)$ exists, it reveals a lot about the relationship of f and g:

$$\lim_{n\to\infty} f/g \ne 0, \infty \Rightarrow f = \Theta(g) \qquad \lim_{n\to\infty} f/g = 1 \Rightarrow f \sim g$$
$$\lim_{n\to\infty} f/g \ne \infty \Rightarrow f = O(g) \qquad \lim_{n\to\infty} f/g = 0 \Rightarrow f = o(g)$$
$$\lim_{n\to\infty} f/g \ne 0 \Rightarrow f = \Omega(g) \qquad \lim_{n\to\infty} f/g = \infty \Rightarrow f = \omega(g)$$

In this context, L'Hospital's Rule is often useful:

$$\text{If } \lim_{n\to\infty} f(n) = \infty \text{ and } \lim_{n\to\infty} g(n) = \infty, \text{ then } \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

Logarithms vs. polynomials vs. exponentials. *Everybody* knows the following two facts:

- polylogarithms grow *slower* than polynomials: for all $a, b > 0$, $(\ln n)^a = o(n^b)$.
- polynomials grow *slower* than exponentials: for all $b, c > 0$, $n^b = o\big((1+c)^n \big)$.