Problems for Recitation 10

$$1 + z + z^2 + \ldots + z^{n-1} = \frac{1 - z^n}{1 - z} \qquad (z \neq 1)$$

$$1 + z + x^2 + \ldots = \frac{1}{1 - z} \qquad (|z| < 1)$$

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + \frac{1}{2})(n+1)}{3}$$

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

Theorem (Taylor's theorem). *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is* $n + 1$ *times differentiable on the interval* $[0, x]$*. Then*

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \ldots + \frac{f^{(n)}(0)x^n}{n!} + \frac{f^{(n+1)}(z)x^{n+1}}{(n+1)!}$$

for some $z \in [0, x]$.

1 Sums and Approximations

Problem 1. Evaluate the following sums.

(a)

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$$

(b)

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots$$

(c)

$$1 + 2 + 4 + 8 + \ldots + 2^{n-1}$$

(d)

$$\sum_{k=n}^{2n} k^2$$

(e)

$$\sum_{i=0}^{n} \sum_{j=0}^{m} 3^{i+j}$$

Problem 2. You've seen this neat trick for evaluating a geometric sum:

$$\begin{aligned} S &= 1 + z + z^2 + \ldots + z^n \\ zS &= z + z^2 + \ldots + z^n + z^{n+1} \\ S - zS &= 1 - z^{n+1} \\ S &= \frac{1 - z^{n+1}}{1 - z} \end{aligned}$$

Use the same approach to find a closed-form expression for this sum:

$$T = 1z + 2z^2 + 3z^3 + \ldots + nz^n$$

Problem 3. Here is a nasty product:

$$\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\left(1+\frac{3}{n^2}\right)\cdots\left(1+\frac{n}{n^2}\right)$$

Remarkably, an expression similar to this one comes up in analyzing the distribution of birthdays. Let's make sense of it.

(a) Give a two-term Taylor approximation for e^x. (Forget about the error term.)

(b) This is probably the most wide-used approximation in computer science. The fact that x appears at "ground level" in the approximation and in the exponent of e^x lets us translate sums into products and vice-versa. Rewrite the product using this approximation.

(c) Now use a standard summation formula to simplify the exponent.

(d) What constant does this approach for large n?

Problem 4. Let's use Taylor's Theorem to find some approximations for the function $\sqrt{1+x}$.

(a) Give a three-term Taylor approximation for $\sqrt{1+x}$.

(b) Sketch the function $\sqrt{1+x}$ and your approximation. How good is the approximation when $x = 8$?

(c) Using this approximation and the fact that $\sqrt{1+x} = \sqrt{x}\sqrt{1+1/x}$, give an approximation for $\sqrt{1+x}$ that is accurate for *large* x

(d) Estimate:

$$\sqrt{1,000,001}$$

to a dozen places beyond the decimal point. You can try to check your answer with a calculator, but you'd better use a good one!