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Notes for Recitation 10

$$1 + z + z^2 + \ldots + z^{n-1} = \frac{1 - z^n}{1 - z} \qquad (z \neq 1)$$
$$1 + z + x^2 + \ldots = \frac{1}{1 - z} \qquad (|z| < 1)$$
$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$
$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+\frac{1}{2})(n+1)}{3}$$
$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

Theorem (Taylor's theorem). *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is $n+1$ times differentiable on the interval $[0, x]$. Then*

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \ldots + \frac{f^{(n)}(0)x^n}{n!} + \frac{f^{(n+1)}(z)x^{n+1}}{(n+1)!}$$

for some $z \in [0, x]$.

1 Sums and Approximations

Problem 1. Evaluate the following sums.

(a)

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$$

Solution. The formula for the sum of an infinite geometric series with ratio $1/2$ gives:

$$\frac{1}{1 - \frac{1}{2}} = 2$$

(b)

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots$$

Solution. The formula for the sum of an infinite geometric series with ratio $-1/2$ gives:

$$\frac{1}{1 - \left(-\frac{1}{2}\right)} = 2/3$$

(c)

$$1 + 2 + 4 + 8 + \ldots + 2^{n-1}$$

Solution. The formula for the sum of a (finite) geometric series with ratio 2 gives:

$$\frac{1 - 2^n}{1 - 2} = 2^n - 1$$

(d)

$$\sum_{k=n}^{2n} k^2$$

Solution.

$$\begin{aligned}\sum_{k=n+1}^{2n} k^2 &= \sum_{k=1}^{2n} k^2 - \sum_{k=1}^{n} k^2 \\ &= \frac{2n(2n + \frac{1}{2})(2n + 1)}{3} - \frac{n(n + \frac{1}{2})(n + 1)}{3}\end{aligned}$$

(e)

$$\sum_{i=0}^{n} \sum_{j=0}^{m} 3^{i+j}$$

Solution.

$$
\begin{aligned}
\sum_{i=0}^{n} \sum_{j=0}^{m} 3^{i+j} &= \sum_{i=0}^{n} \left(3^i \cdot \sum_{j=0}^{m} 3^j \right) \\
&= \left(\sum_{j=0}^{m} 3^j \right) \cdot \left(\sum_{i=0}^{n} 3^i \right) \\
&= \left(\frac{3^{m+1} - 1}{2} \right) \cdot \left(\frac{3^{n+1} - 1}{2} \right)
\end{aligned}
$$

Problem 2. You've seen this neat trick for evaluating a geometric sum:

$$\begin{aligned} S &= 1 + z + z^2 + \ldots + z^n \\ zS &= z + z^2 + \ldots + z^n + z^{n+1} \\ S - zS &= 1 - z^{n+1} \\ S &= \frac{1 - z^{n+1}}{1 - z} \end{aligned}$$

Use the same approach to find a closed-form expression for this sum:

$$T = 1z + 2z^2 + 3z^3 + \ldots + nz^n$$

Solution.

$$\begin{aligned} zT &= 1z^2 + 2z^3 + 3z^4 + \ldots + nz^{n+1} \\ T - zT &= z + z^2 + z^3 + \ldots + z^n - nz^{n+1} \\ &= \frac{1 - z^{n+1}}{1 - z} - 1 - nz^{n+1} \\ T &= \frac{1 - z^{n+1}}{(1 - z)^2} - \frac{1 + nz^{n+1}}{1 - z} \end{aligned}$$

Problem 3. Here is a nasty product:

$$\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\left(1+\frac{3}{n^2}\right)\cdots\left(1+\frac{n}{n^2}\right)$$

Remarkably, an expression similar to this one comes up in analyzing the distribution of birthdays. Let's make sense of it.

(a) Give a two-term Taylor approximation for e^x. (Forget about the error term.)

Solution.

$$e^x \approx 1 + x$$

(b) This is probably the most wide-used approximation in computer science. The fact that x appears at "ground level" in the approximation and in the exponent of e^x lets us translate sums into products and vice-versa. Rewrite the product using this approximation.

Solution.

$$e^{1/n^2} \cdot e^{2/n^2} \cdot e^{3/n^2} \cdot \cdots \cdot e^{n/n^2} = e^{\frac{1+2+\ldots+n}{n^2}}$$

(c) Now use a standard summation formula to simplify the exponent.

Solution. The formula $1 + 2 + 3 + \ldots + n = n(n+1)/2$ gives:

$$e^{n(n+1)/(2n^2)} = e^{1/2+1/2n}$$

(d) What constant does this approach for large n?

Solution. $\sqrt{e}$

Problem 4. Let's use Taylor's Theorem to find some approximations for the function $\sqrt{1+x}$.

(a) Give a three-term Taylor approximation for $\sqrt{1+x}$.

Solution. First, we compute two derivatives:

$$f'(x) = \frac{1}{2\sqrt{1+x}}$$
$$f''(x) = -\frac{1}{4(1+x)^{3/2}}$$

Now we plug into Taylor's theorem:

$$f(x) \approx f(0) + xf'(0)$$
$$1 + \frac{x}{2} - \frac{x^2}{8}$$

(b) Sketch the function $\sqrt{1+x}$ and your approximation. How good is the approximation when $x = 8$?

Solution. The approximation is pretty bad when $x = 8$. The actual value is 3, but the approximation is -3.

(c) Using this approximation and the fact that $\sqrt{1+x} = \sqrt{x}\sqrt{1+1/x}$, give an approximation for $\sqrt{1+x}$ that is accurate for *large* x

Solution.

$$\begin{aligned}\sqrt{1+x} &= \sqrt{x}\sqrt{1+1/x} \\ &\approx \sqrt{x}\left(1 + \frac{1}{2x} + \frac{1}{8x^2}\right)\end{aligned}$$

(d) Estimate:

$$\sqrt{1,000,001}$$

to a dozen places beyond the decimal point. You can try to check your answer with a calculator, but you'd better use a good one!

Solution.

$$\begin{aligned}\sqrt{1,000,001} &\approx 1000 \cdot \left(1 + \frac{1}{2\cdot 10^6} + \frac{1}{8\cdot 10^{12}}\right) \\ &= 1000.000500000125\end{aligned}$$