Problem Set 6 Solutions

Due: Monday, March 28 at 9 PM in Room 32-044

Problem 1. Sammy the Shark is a financial service provider who offers loans on the following terms.

- Sammy loans a client m dollars in the morning. This puts the client m dollars in debt to Sammy.
- Each evening, Sammy first charges a "service fee", which increases the client's debt by f dollars, and then Sammy charges interest, which multiplies the debt by a factor of p. For example, if Sammy's interest rate were a modest 5% per day, then p would be 1.05.

(a) What is the client's debt at the end of the first day?

Solution. At the end of the first day, the client owes Sammy $(m + f)p = mp + fp$ dollars.

(b) What is the client's debt at the end of the second day?

Solution. $((m + f)p + f)p = mp^2 + fp^2 + fp$

(c) Write a formula for the client's debt after d days and find an equivalent closed form.

Solution. The client's debt after three days is

$$(((m + f)p + f)p + f)p = mp^3 + fp^3 + fp^2 + fp.$$

Generalizing from this pattern, the client owes

$$mp^d + \sum_{k=1}^{d} fp^k$$

dollars after d days. Applying the formula for a geometric sum gives:

$$mp^d + f \cdot \left(\frac{p^{d+1} - 1}{p - 1} - 1 \right)$$

Problem 2. Find closed-form expressions equal to the following sums. Show your work.

(a)

$$\sum_{i=0}^{n} \frac{9^i - 7^i}{11^i}$$

Solution. Split the expression into two geometric series and then apply the formula for the sum of a geometric series.

$$\begin{aligned}
\sum_{i=0}^{n} \frac{9^i - 7^i}{11^i} &= \sum_{i=0}^{n} \left(\frac{9}{11}\right)^i - \sum_{i=0}^{n} \left(\frac{7}{11}\right)^i \\
&= \frac{1 - \left(\frac{9}{11}\right)^{n+1}}{1 - \frac{9}{11}} - \frac{1 - \left(\frac{7}{11}\right)^{n+1}}{1 - \frac{7}{11}} \\
&= -\frac{11}{2} \cdot \left(\frac{9}{11}\right)^{n+1} + \frac{11}{4} \cdot \left(\frac{7}{11}\right)^{n+1} + \frac{11}{4}
\end{aligned}$$

(b)

$$\prod_{i=1}^{n} 3^{4i+5}$$

Solution. Taking the logarithm reduces this product to an easy sum.

$$\begin{aligned}
\prod_{i=1}^{n} 3^{4i+5} &= 3^{\log_3\left(\prod_{i=1}^{n} 3^{4i+5}\right)} \\
&= 3^{\sum_{i=1}^{n} 4i+5} \\
&= 3^{2n(n+1)+5n}
\end{aligned}$$

(c)

$$\sum_{j=1}^{n} \sum_{i=0}^{\infty} j^{5/3} \cdot \left(1 - \frac{1}{2j^{1/3}}\right)^i$$

Solution. This fearsome-looking sum is a paper tiger; we just apply the formula for the sum of a geometric series followed by the formula for the sum of an arithmetic series.

$$\begin{aligned}
\sum_{j=1}^{n} \sum_{i=0}^{\infty} j^{5/3} \cdot \left(1 - \frac{1}{2j^{1/3}}\right)^i &= \sum_{j=1}^{n} j^{5/3} \cdot \frac{1}{1 - \left(1 - \frac{1}{2j^{1/3}}\right)} \\
&= \sum_{j=1}^{n} 2j^2 \\
&= \frac{2n(n + \frac{1}{2})(n+1)}{3}
\end{aligned}$$

Problem 3. There is a bug on the edge of a 1-meter rug. The bug wants to cross to the other side of the rug. It crawls at 1 cm per second. However, at the end of each second, a malicious first-grader named Mildred Anderson *stretches* the rug by 1 meter. Assume that her action is instantaneous and the rug stretches uniformly. Thus, here's what happens in the first few seconds:

- The bug walks 1 cm in the first second, so 99 cm remain ahead.
- Mildred stretches the rug by 1 meter, which doubles its length. So now there are 2 cm behind the bug and 198 cm ahead.
- The bug walks another 1 cm in the next second, leaving 3 cm behind and 197 cm ahead.
- Then Mildred strikes, stretching the rug from 2 meters to 3 meters. So there are now $3 \cdot (3/2) = 4.5$ cm behind the bug and $197 \cdot (3/2) = 295.5$ cm ahead.
- The bug walks another 1 cm in the third second, and so on.

Your job is to determine this poor bug's fate.

(a) During second i, what *fraction* of the rug does the bug cross?

Solution. During second i, the length of the rug is $100i$ cm and the bug crosses 1 cm. Therefore, the fraction that the bug crosses is $1/100i$.

(b) Over the first n seconds, what fraction of the rug does the bug cross altogether?

Solution. The bug crosses $1/100$ of the rug in the first second, $1/200$ in the second, $1/300$ in the third, and so forth. Thus, over the first n seconds, the fraction crossed by the bug is:

$$\sum_{k=1}^{n} \frac{1}{100k} = H_n/100$$

(This formula is valid only until the bug reaches the far side of the rug.)

(c) Approximately how many seconds does the bug need to cross the entire rug?

Solution. The bug arrives at the far side when the fraction it has crossed reaches 1. This occurs when n, the number of seconds elapsed, is sufficiently large that $H_n/100 \geq 1$. Now H_n is approximately $\ln n$, so the bug arrives about when:

$$\begin{aligned} \frac{\ln n}{100} &\geq 1 \\ \ln n &\geq 100 \\ n &\geq e^{100} \approx 10^{43} \text{ seconds} \end{aligned}$$

Problem 4. Use integration to find lower and upper bounds on the following infinite sum that differ by at most 0.1. Show your work.

$$S = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

To achieve this accuracy, add up the first few terms explicitly and then use integration to bound all remaining terms.

Solution. The sum of the first three terms is:

$$s = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} = \frac{49}{36}$$

An upper bound on the remaining terms is:

$$\int_3^\infty \frac{1}{x^2}\, dx = \frac{1}{3}$$

And a lower bound is:

$$\int_3^\infty \frac{1}{(x+1)^2}\, dx = \frac{1}{4}$$

Overall, we have:

$$\frac{58}{36} \leq \frac{49}{36} + \frac{1}{4} \leq S \leq \frac{49}{36} + \frac{1}{3} = \frac{61}{36}$$

These bounds differ by $1/12 < 0.1$. The actual value of the sum is $\pi^2/6$, though the proof is not easy.

Problem 5. A seasoned MIT undergraduate can:

- Complete a problem set in 2 days.
- Write a paper in 2 days.
- Take a 2-day road trip.
- Study for an exam in 1 day.
- Play foosball for an entire day.

An *n-day schedule* is a sequence of activities that require a total of n days. For example, here are three possible 7-day schedules:

pset, paper, pset, foosball
paper, study, foosball, pset, study
road trip, road trip, road trip, study

(a) Express the number of possible n-day schedules using a recurrence equation and sufficient base cases.

Solution.

$$S(0) = 1,$$
$$S(1) = 2.$$

Any schedule for $n > 1$ days ends with one of 3 possible 2-day activities or one of 2 possible 1-day activities. So

$$S(n) = 2S(n-1) + 3S(n-2) \qquad \text{for } n > 1.$$

(b) Find a closed-form expression for the number of possible n-day schedules by solving the recurrence.

Solution. The characteristic polynomial for this linear homogeneous recurrence is $x^2 - 2x - 3 = (x+1)(x-3)$. Hence the solution is of the form $S(n) = a(-1)^n + b3^n$. Letting $n = 0$, we conclude that $a+b = 1$, and letting $n = 1$, we conclude $-a+3b = 2$, so $b = 3/4$, $a = 1/4$, and the solution is:

$$S(n) = \frac{3^{n+1} + (-1)^n}{4}.$$

Problem 6. Find a closed-form expression for $T(n)$, which is defined by the following recurrence:

$$T(0) = 0$$
$$T(1) = 1$$
$$T(n) = 5T(n-1) - 6T(n-2) + 6 \qquad \text{for all } n \geq 2$$

Solution. The characteristic equation is $x^2 - 5x + 6 = 0$, which has roots $x = 2$ and $x = 3$. Thus, the homogenous solution is:

$$T(n) = A \cdot 2^n + B \cdot 3^n$$

For a particular solution, let's first guess $T(n) = c$:

$$c = 5c - 6c + 6$$
$$\Rightarrow c = 3$$

Our guess was correct; $T(n) = 3$ is a particular solution. Adding this to the homogenous solution gives the general solution:

$$T(n) = A \cdot 2^n + B \cdot 3^n + 3$$

Substituting $n = 0$ and $n = 1$ gives:

$$0 = A + B + 3$$
$$1 = 2A + 3B + 3$$

Solving this system gives $A = -7$ and $B = 4$. Therefore:

$$T(n) = -7 \cdot 2^n + 4 \cdot 3^n + 3$$

Problem 7. Determine which of these choices

$$\Theta(n), \quad \Theta(n^2 \log n), \quad \Theta(n^2), \quad \Theta(1), \quad \Theta(2^n), \quad \Theta(2^{n \ln n}), \quad \text{none of these}$$

describes each function's asymptotic behavior. Proofs are not required, but briefly explain your answers.

(a)

$$n + \ln n + (\ln n)^2$$

Solution. Both $n > \ln n$ and $n > (\ln n)^2$ hold for all sufficiently large n. Thus, for all sufficiently large n:

$$n < n + \ln n + (\ln n)^2 < n + n + n$$

So $n + \ln n + (\ln n)^2 = \Theta(n)$.

(b)

$$\frac{n^2 + 2n - 3}{n^2 - 7}$$

Solution. Observe that:

$$\lim_{n \to \infty} \frac{n^2 + 2n - 3}{n^2 - 7} = 1$$

This means, that for all sufficiently large n, the fraction lies, for example, between, 0.99 and 1.01 and is therefore $\Theta(1)$.

(c)

$$\sum_{i=0}^{n} 2^{2i+1}$$

Solution. Geometric sums are dominated by their largest term, which is $2^{2n+1} = 2 \cdot 4^n$. This is $\Theta(4^n)$, which does not appear in the list provided.

(d)

$$\ln(n^2!)$$

Solution. By Stirling's formula:

$$n^2! \sim \sqrt{2\pi n^2} \left(\frac{n^2}{e} \right)^{n^2}$$

Taking logarithms gives:

$$\begin{aligned}\ln(n^2!) &\sim \ln(\sqrt{2\pi n^2}\left(\frac{n^2}{e}\right)^{n^2}) \\ &= \ln(\sqrt{2\pi n^2}) + \ln\left(\frac{n^2}{e}\right)^{n^2}\end{aligned}$$

The first term is tiny compared to the second, which we can rewrite as:

$$\ln\left(\frac{n^2}{e}\right)^{n^2} = n^2 \ln\left(\frac{n^2}{e}\right) = \Theta(n^2 \ln n)$$

(e)

$$\sum_{k=1}^{n} k\left(1 - \frac{1}{2^k}\right)$$

Solution. The expression in parentheses is always at least $1/2$ and at most 1. Thus, we have the bounds:

$$\frac{1}{2}\sum_{k=1}^{n} k \le \sum_{k=1}^{n} k\left(1 - \frac{1}{2^k}\right) \le \sum_{k=1}^{n} k$$

Since the first expression and the last are both $\Theta(n^2)$, so is the one in the middle.

Problem 8. A *triangular number* is an integer n of the form

$$n = 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$$

where k is a positive integer.

(a) Describe a solution to the four-peg Towers of Hanoi puzzle with $k(k+1)/2$ disks that requires T_k moves, where:

$$\begin{aligned}T_1 &= 1 \\ T_k &= 2T_{k-1} + 2^k - 1\end{aligned}$$

(b) Find a closed form expression equal to T_k.

Solution. This is an inhomogenous linear equation. Let's begin by trying to find a particular solution. There is both an exponential term (2^k) and a constant term, so we might guess something of the form $a2^k + c$:

$$\begin{aligned}a2^k + c &= 2(a2^{k-1} + c) + 2^k - 1 \\ &= (a+1)2^k + 2c - 1 \\ \Rightarrow 0 &= 2^k + (c-1)\end{aligned}$$

Evidently, the constant term is $c = 1$, but the exponential part is more complicated. Our recipe says we should next try a particular solution of the form $a2^k + bk2^k + 1$:

$$\begin{aligned} a2^k + bk2^k + 1 &= 2(a2^{k-1} + b(k-1)2^{k-1} + 1) + 2^k - 1 \\ &= (a - b + 1)2^k + bk2^k - 1 \end{aligned}$$

Equating the coefficients of the 2^k terms gives $a = a - b + 1$, which implies $b = 1$. Thus, $a2^k + k2^k + 1$ is a particular solution for all a. As long as we have this degree of freedom, we might as well choose a so this solution is consistent with the boundary condition $T_1 = 1$ and be done:

$$a2^1 + 1 \cdot 2^1 + 1 = 1 \quad \Rightarrow \quad a = -1$$

Therefore, the solution to the recurrence is $T_k = (k-1)2^k + 1$.

(c) Approximately how many moves are required to solve the four-peg, n-disk Towers of Hanoi puzzle as a function of n? Assume n is a triangular number. (For style points, make correct use of asymptotic notation.)

Solution. We have $k = \frac{1}{2}(\sqrt{8n+1} - 1) = \sqrt{2n} + O(1)$. So the number of moves required is $\Theta(\sqrt{n}2^{\sqrt{2n}})$.