

Problem Set 11 Solutions

Due: 5PM on Friday, May 6

This is a mini-problem set. The first problem reviews basic facts about expectation. The second and third are typical final exam questions.

Problem 1. Answer the following questions about expectation.

- (a) There are several equivalent definitions of the expectation of a random variable. If R is a random variable over sample space S , then we can compute $\text{Ex}(R)$ by summing over individual outcomes or by summing over values in the range of R . Write down these two equivalent definitions of $\text{Ex}(R)$.

Solution.

$$\text{Ex}(R) = \sum_{w \in S} R(w) \Pr(w) = \sum_{v \in \text{Range}(R)} v \cdot \Pr(R = v)$$

- (b) Give another expression for $\text{Ex}(R)$ that holds when R is a natural-valued random variable.

Solution.

$$\text{Ex}(R) = \sum_{k=0}^{\infty} \Pr(R > k)$$

- (c) Give a simple expression for $\text{Ex}(R)$ that is valid when R is an indicator random variable.

Solution.

$$\text{Ex}(R) = \Pr(R = 1)$$

- (d) The expectation of a random variable can often be computed from expectations of simpler random variables. Rewrite each of the expressions below in terms of $\text{Ex}(R)$ and $\text{Ex}(S)$. Note any conditions that R and S must satisfy in order for your equations to hold. Here c is a constant.

$$\text{Ex}(cR) \quad \text{Ex}(R + S) \quad \text{Ex}(R \cdot S)$$

Solution.

$$\begin{aligned}\text{Ex}(cR) &= c \text{Ex}(R) \\ \text{Ex}(R + S) &= \text{Ex}(R) + \text{Ex}(S) \\ \text{Ex}(R \cdot S) &= \text{Ex}(R) \cdot \text{Ex}(S)\end{aligned}$$

The third equation holds only if R and S are independent.

- (e) The expected value of a random variable R given that some event E occurs is denoted $\text{Ex}(R | E)$. Write down two equivalent expressions for $\text{Ex}(R | E)$ based on your answers to part (a).

Solution.

$$\text{Ex}(R | E) = \sum_{w \in S} R(w) \Pr(w | E) = \sum_{v \in \text{Range}(R)} v \cdot \Pr(R = v | E)$$

- (f) Sometimes the job of computing $\text{Ex}(R)$ is best broken down into cases. Let E_1, \dots, E_n be events that partition the sample space. Suppose you can compute $\text{Ex}(R | E_k)$ and $\Pr(E_k)$ for all k . How do you then compute $\text{Ex}(R)$?

Solution.

$$\text{Ex}(R) = \sum_{k=1}^n \text{Ex}(R | E_k) \Pr(E_k) \quad (1)$$

- (g) Many problems involve a sequence of independent trials, each of which succeeds with probability p . What is the expected number of trials needed to obtain one success?

Solution. $1/p$

Problem 2. MIT students sometimes delay laundry for a few days. Assume all random values described below are mutually independent.

- (a) A *busy* student must complete 3 problem sets before doing laundry. Each problem set requires 1 day with probability $2/3$ and 2 days with probability $1/3$. Let B be the number of days a busy student delays laundry. What is $\text{Ex}(B)$?

Example: If the first problem set requires 1 day and the second and third problem sets each require 2 days, then the student delays for $B = 5$ days.

Solution. The expected time to complete a problem set is:

$$1 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3} = \frac{4}{3}$$

Therefore, the expected time to complete all three problem sets is:

$$\begin{aligned}\text{Ex}(B) &= \text{Ex}(\text{pset1}) + \text{Ex}(\text{pset2}) + \text{Ex}(\text{pset3}) \\ &= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} \\ &= 4\end{aligned}$$

(b) A *relaxed* student rolls a fair, 6-sided die in the morning. If he rolls a 1, then he does his laundry immediately (with zero days of delay). Otherwise, he delays for one day and repeats the experiment the following morning. Let R be the number of days a relaxed student delays laundry. What is $\text{Ex}(R)$?

Example: If the student rolls a 2 the first morning, a 5 the second morning, and a 1 the third morning, then he delays for $R = 2$ days.

Solution. If we regard doing laundry as a failure, then the mean time to failure is $1/(1/6) = 6$. However, this counts the day laundry is done, so the number of days delay is $6 - 1 = 5$. Alternatively, we could derive the answer as follows:

$$\begin{aligned}\text{Ex}(R) &= \sum_{k=0}^{\infty} \Pr(R > k) \\ &= \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \dots \\ &= \frac{5}{6} \cdot \left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \dots\right) \\ &= \frac{5}{6} \cdot \frac{1}{1 - 5/6} \\ &= 5\end{aligned}$$

(c) Before doing laundry, an *unlucky* student must recover from illness for a number of days equal to the product of the numbers rolled on two fair, 6-sided dice. Let U be the expected number of days an unlucky student delays laundry. What is $\text{Ex}(U)$?

Example: If the rolls are 5 and 3, then the student delays for $U = 15$ days.

Solution. Let D_1 and D_2 be the two die rolls. Recall that a die roll has expectation $7/2$. Thus:

$$\begin{aligned}\text{Ex}(U) &= \text{Ex}(D_1 \cdot D_2) \\ &= \text{Ex}(D_1) \cdot \text{Ex}(D_2) \\ &= \frac{7}{2} \cdot \frac{7}{2} \\ &= \frac{49}{4}\end{aligned}$$

(d) A student is *busy* with probability $1/2$, *relaxed* with probability $1/3$, and *unlucky* with probability $1/6$. Let D be the number of days the student delays laundry. What is $\text{Ex}(D)$?

Solution.

$$\text{Ex}(D) = \frac{1}{2} \text{Ex}(B) + \frac{1}{3} \text{Ex}(R) + \frac{1}{6} \text{Ex}(U)$$

Problem 3. I have twelve cards:

1	1	2	2	3	3	4	4	5	5	6	6
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I shuffle them and deal them in a row. For example, I might get:

1	2	3	3	4	6	1	4	5	5	2	6
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What is the expected number of adjacent pairs with the same value? In the example, there are two adjacent pairs with the same value, the 3's and the 5's.

Solution. Consider an adjacent pair. The left card matches only one of the other 11 cards, which is equally likely to be in any of the 11 other positions. Therefore, the probability that an adjacent pair matches is $1/11$. Since there are 11 adjacent pairs, the expected number of matches is $11 \cdot 1/11 = 1$ by linearity of expectation.