Sums and Approximations

When you analyze the running time of an algorithm, the probability some procedure succeeds, or the behavior of a load-balancing or communications scheme, you'll *rarely* get a simple answer. The world is not so kind. More likely, you'll end up with a complicated sum:

$$\sum_{k=1}^{n} \frac{1}{k + \sqrt{k}}$$

Or a nasty product:

$$\left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2}{n^2}\right)\left(1 + \frac{3}{n^2}\right)\cdots\left(1 + \frac{n}{n^2}\right)$$

Or you might get an answer that is just tad too complicated to grasp intuitively:

$$\left(1 + \frac{72/n}{100}\right)^n$$

And these examples are relatively benign!

So the ability to cope with such messy mathematical expressions is an important skill in computer science— and in many other areas of science and engineering. This might not seem glorious, but people who can cut monstrous expressions down to size in moments can seem *pretty amazing* to the uninitiated. This week, we'll equip you with the most useful tricks of the trade.

1   The Value of an Annuity

Would you prefer a million dollars today or $20,000 a year for the next fifty years?

This is a question about the value of an **annuity**, a financial instrument that pays out a fixed amount of money at the beginning of every year for some specified number of years. In particular, an n-year, m-payment annuity pays m dollars at the start of each year for n years. In some cases, n is finite, but not always. Examples include lottery payouts, student loans, and home mortgages. There are even Wall Street people who specialize in trading annuities.

For many reasons, $20,000 a year for 50 years is worth much less than a million dollars right now. For example, consider the last $20,000 installment. If you had that $20,000 right now, then you could start earning interest, invest the money in the stock market,

or just buy something fun. However, if you don't get the $20,000 for another 50 years, then someone else is earning all the interest or investment profit. Furthermore, prices are likely to gradually rise over the next 50 years, so you when you finally get the money, you won't be able to buy as much. Finally, people only live so long; if you were 60 years old, a payout 50 years in the future would be worth next to nothing!

But what if your choice were between *$40,000* a year for 50 years and a million dollars today? Now which is better? What is an annuity is actually worth?

1.1 The Future Value of Money

In order to address such questions, we have to make an assumption about the future value of money. Let's put most of the complications aside and think about this from a simple-minded perspective. The average rate of inflation in the United States from 1980 to 2004 was about $p = 3.5\%$ per year. This means that the price of a selection of basic goods increases by about 3.5% each year. If this trend continues, then goods costing $100 today will cost:

$$\$100(1+p) = \$103.50 \text{ in 1 year}$$
$$\$100(1+p)^2 = \$107.12 \text{ in 2 year}$$
$$\ldots$$
$$\$100(1+p)^k \text{ in k years}$$

Looked at another way, k years from now, $100 will have the buying power of just $100/(1+p)^k$ dollars today.

Now we can work out the value of an annuity that pays m dollars at the start of each year for the next n years:

payments	current value
$m today	m
$m in 1 year	$\dfrac{m}{1+p}$
$m in 2 years	$\dfrac{m}{(1+p)^2}$
$\ldots$	$\ldots$
$m in $n-1$ years	$\dfrac{m}{(1+p)^{n-1}}$

$$\text{Total current value:} \quad V = \sum_{k=0}^{n-1} \frac{m}{(1+p)^k}$$

So to compute the value of the annuity, we need only evaluate this sum. We *could* plug in values for m, n, and p, compute each term explicitly, and then add them all up. However, this particular sum has an equivalent "closed form" that makes the job easier. In general, a **closed form** is a mathematical expression that can be evaluated with a fixed number of basic operations (addition, multiplication, exponentiation, etc.) In contrast, evaluating the sum above requires a number of operations proportional to n.

1.2   A Geometric Sum

Our goal is to find a closed form equivalent to the summation:

$$V = \sum_{k=0}^{n-1} \frac{m}{(1+p)^k} = m + \frac{m}{1+p} + \frac{m}{(1+p)^2} + \ldots + \frac{m}{(1+p)^{k-1}}$$

This is a **geometric sum**, which means that consecutive terms all have the same ratio. In particular, the second term is $1/(1+p)$ times the first, the third is $1/(1+p)$ times the second, and so forth. And we've already encountered a theorem about geometric sums:

Theorem 1. *For all $n \geq 1$ and all $z \neq 1$:*

$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z}$$

This theorem can be proved by induction, but that proof gives no hint how the formula might be found in the first place. Here is a more insightful derivation based on the **perturbation method**. First, we let S equal the value of the sum and then "perturb" it by multiplying by z.

$$
\begin{array}{rclclclclclcl}
S &=& 1 &+& z &+& z^2 &+& \ldots &+& z^{n-1} & & \\
zS &=& & & z &+& z^2 &+& \ldots &+& z^{n-1} &+& z^n
\end{array}
$$

The difference between the original sum and the perturbed sum is not so great, because there is massive cancellation on the right side:

$$S - zS = 1 - z^n$$

Now solving for S gives the expression in Theorem 1:

$$S = \frac{1-z^n}{1-z}$$

You can derive a passable number of summation formulas by mimicking the approach used above. We'll look at some other methods for evaluting sums shortly.

1.3 Return of the Annuity Problem

Now we can solve the annuity pricing problem! The value of an annuity that pays m dollars at the start of each year for n years is:

$$
\begin{aligned}
V &= \sum_{k=0}^{n-1} \frac{m}{(1+k)^p} \\
&= m \sum_{k=0}^{n-1} z^k \qquad \left(\text{where } z = \frac{1}{1+p}\right) \\
&= m \cdot \frac{1 - z^n}{1 - z} \\
&= m \cdot \frac{1 - \left(\frac{1}{1+p}\right)^n}{1 - \left(\frac{1}{1+p}\right)}
\end{aligned}
$$

We apply Theorem 1 on the second line, and undo the the earlier substitution $z = 1/(1+p)$ on the last line.

The last expression is a closed form; it can be evaluated with a fixed number of basic operations. For example, what is the real value of a winning lottery ticket that pays $\$40,000$ per year for 50 years? Plugging in $m = \$40,000$, $n = 50$, and $p = 0.035$ gives $V \approx \$971,063$. You'd be better off taking the million dollars today!

1.4 Infinite Sums

All right, would you prefer a million dollars today or $\$40,000$ a year *forever*? This might seem like an easy choice— when infinite money is on offer, why worry about inflation?

This is a question about an **infinite sum**. In general, the value of an infinite sum is defined as the limit of a finite sum as the number of terms goes to infinity:

$$
\sum_{k=0}^{\infty} z_k \qquad \text{means} \qquad \lim_{n \to \infty} \sum_{k=0}^{n} z_k
$$

So the value of an annuity with an infinite number of payments is given by our previous answer in the limit as n goes to infinity:

$$
\begin{aligned}
V &= \lim_{n \to \infty} m \cdot \frac{1 - \left(\frac{1}{1+p}\right)^n}{1 - \left(\frac{1}{1+p}\right)} \\
&= m \cdot \frac{1}{1 - \left(\frac{1}{1+p}\right)} \\
&= m \cdot \frac{1+p}{p}
\end{aligned}
$$

In the second step, notice that the $1/(1 + p)^n$ term in the numerator goes to zero in the limit. The third equation follows by simplifying.

Plugging in $m = \$40,000$ and $p = 0.035$ into this formula gives $V \approx \$1,182,857$. The value of money drops off so fast that even an *infinite* number of payments are worth on a bit over a million dollars. In fact, the total value of all payments beyond the 50th is only about $200,000!

More generally, we can get a closed form for infinite geometric sums from Theorem 1 by taking a limit.

Corollary 2. *If $|z| < 1$, then:*

$$\sum_{i=0}^{\infty} z^i = \frac{1}{1 - z}$$

Proof.

$$\sum_{i=0}^{\infty} z^i = \lim_{n \to \infty} \sum_{i=0}^{n-1} z^i$$
$$= \lim_{n \to \infty} \frac{1 - z^n}{1 - z}$$
$$= \frac{1}{1 - z}$$

The first equation uses the definition of an infinite limit, and the second uses Theorem 1. In the limit, the term z^{n+1} in the numerator vanishes since $|z| < 1$. $\square$

We now have closed forms for both finite and infinite geometric series. Some examples are given below. In each case, the solution follows immediately from either Theorem 1 (for finite series) or Corollary 2 (for infinite series).

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} (1/2)^i = \frac{1}{1 - (1/2)} = 2$$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} (-1/2)^i = \frac{1}{1 - (-1/2)} = \frac{2}{3}$$

$$1 + 2 + 4 + 8 + \ldots + 2^{n-1} = \sum_{i=0}^{n-1} 2^i = \frac{1 - 2^n}{1 - 2} = 2^n - 1$$

$$1 + 3 + 9 + 27 + \ldots + 3^{n-1} = \sum_{i=0}^{n-1} 3^i = \frac{1 - 3^n}{1 - 3} = \frac{3^n - 1}{2}$$

Here is a good rule of thumb: *the sum of a geometric series is approximately equal to the term with greatest absolute value.* In the first two examples, the largest term is equal to 1 and the sums are 2 and 2/3, which are both relatively close to 1. In the third example, the sum is about twice the largest term. In the final example, the largest term is 3^{n-1} and the sum is $(3^n - 1)/2$, which is 1.5 times greater.
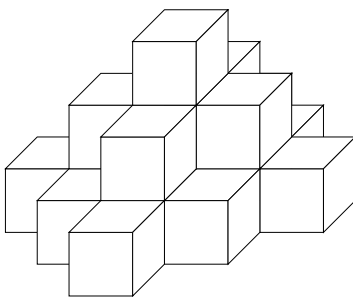
2   Sums of Powers

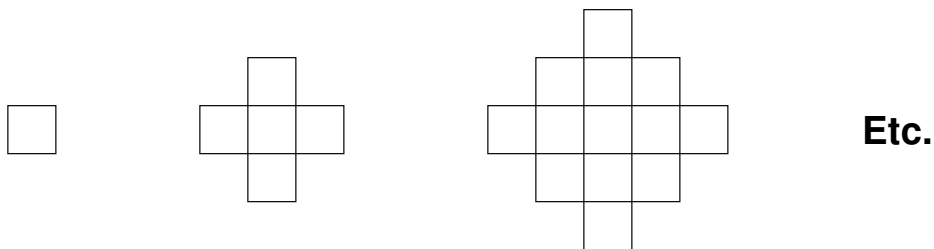Pharaoh Aha I decides to build a "pyramid" in his own honor consisting of a single block:
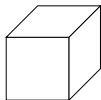
His successor, Aha II, trumps him by building a larger pyramid:
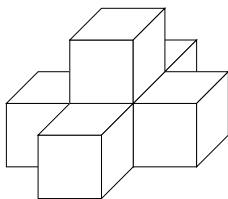
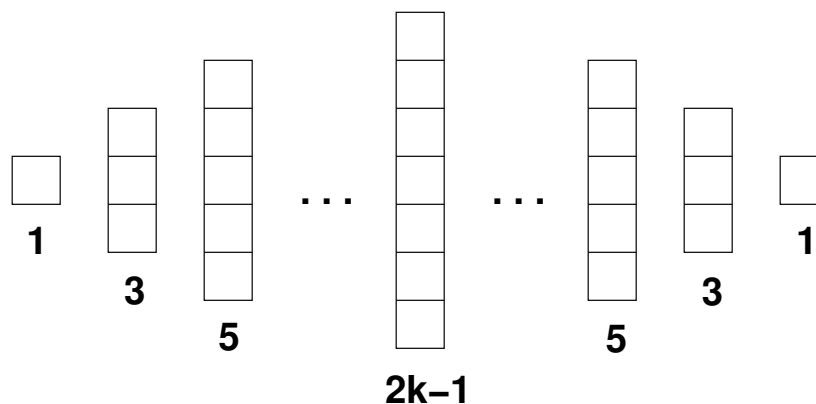Not to be outdone, Aha III builds a still-larger pyramid:

If this madness continues, how many blocks will Pharoah Aha n require? Let's break the problem down by chopping the n-th pyramid into n horizontal slabs:

Etc.

Therefore, we have:

$$\text{\# blocks in } n\text{-th pyramid} = \sum_{k=1}^{n} \text{\# blocks in } k\text{-th slab}$$

Similarly, we can slice each slab into columns:

So the number of blocks in the k-th slab is:

$$1 + 3 + 5 + \ldots + (2k - 1) + \ldots + 5 + 3 + 1 = (2k - 1) + 2 \cdot \sum_{j=1}^{k-1} 2j - 1$$

Plugging this into the previous sum gives the total number of blocks required by Pharaoh Aha-n:

$$\text{\# blocks in } n\text{-th pyramid} = \sum_{k=1}^{n} \left((2k - 1) + 2 \cdot \sum_{j=1}^{k-1} 2j - 1 \right)$$

This sum is pretty nasty. And the bad news is that similar sums arise when analyzing the running time of a program with nested loops.

2.1 Evaluating the Sum

Sums of small powers are actually easy to handle, because there are simple closed-form equivalents:

Lemma 3 (Sums of Powers).

$$1 + 2 + 3 + \ldots + m = \frac{m(m + 1)}{2}$$

$$1^2 + 2^2 + 3^2 + \ldots + m^2 = \frac{m(m + \frac{1}{2})(m + 1)}{3}$$

$$1^3 + 2^3 + 3^3 + \ldots + m^3 = \frac{m^2(m + 1)^2}{4}$$

These formulas can all be proved by routine induction arguments. For now, let's apply them to to find a closed-form expression for the number of blocks in a pyramid. First,

let's tackle the inner sum:

$$\sum_{j=1}^{k-1} 2j - 1 = 2 \left(\sum_{j=1}^{k-1} j \right) - \left(\sum_{j=1}^{k-1} 1 \right)$$

$$= 2 \cdot \frac{(k-1)k}{2} - (k-1)$$

$$= (k-1)^2$$

On the first line, we're using a standard maneuver: *break up the sum of a polynomial into one sum for each term*. In this case, we break the sum of $2j - 1$ into one sum involving j and one sum for the constant. This allows us to apply the standard formulas above, and the rest is simplification. We'll use the same trick to finish off the whole pyramid problem, though now we have to sum values of a quadratic polynomial:

$$\# \text{blocks in } n\text{-th pyramid} = \sum_{k=1}^{n} \left((2k - 1) + 2 \cdot \sum_{j=1}^{k-1} 2j - 1 \right)$$

$$= \sum_{k=1}^{n} (2k - 1) + 2(k - 1)^2$$

$$= \sum_{k=1}^{n} 2k^2 - 2k + 1$$

$$= 2 \sum_{k=1}^{n} k^2 - 2 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= 2 \cdot \frac{n(n + \frac{1}{2})(n + 1)}{3} - 2 \cdot \frac{n(n + 1)}{2} + n$$

$$= \frac{2n^3 + n}{3}$$

We're done! But let's check a couple easy cases to make sure we made no algebra mistakes. This formula says that a pyramid of size 1 contains $(2 \cdot 1^3 + 1)/3 = 1$ block and a pyramid of size 2 contains $(2 \cdot 2^3 + 2)/3 = 6$ blocks— both of which are correct.

2.2   Where Do the Formulas Come From?

Sure, we can prove all the summation formulas in Lemma 3 by induction, but where did the expressions on the right come from? How on earth would we even guess an analogous summation formula for, say, fourth powers?

Here is systematic way to generate a good guess. Remember that sums are the discrete cousins of integrals. So we might guess that the sum of a degree-k polynomial is a degree-$(k + 1)$ polynomial, just as if we were integrating. Based on this observation, we might

guess that

$$\sum_{i=1}^{n} i^2 = an^3 + bn^2 + cn + d$$

for some constants a, b, c, and d. All that remains is to determine these constants. We can do this by plugging a few values for n into the equation above. Each value gives a linear equation in a, b, c, and d. For example:

$$
\begin{aligned}
n = 0 &\quad\Rightarrow\quad 0 = d \\
n = 1 &\quad\Rightarrow\quad 1 = a + b + c + d \\
n = 2 &\quad\Rightarrow\quad 5 = 8a + 4b + 2c + d \\
n = 3 &\quad\Rightarrow\quad 14 = 27a + 9b + 3c + d
\end{aligned}
$$

We now have four equations in four unknowns. Solving this system gives $a = 1/3$, $b = 1/2$, $c = 1/6$, and $d = 0$, so it is tempting to conclude that:

$$
\begin{aligned}
\sum_{i=1}^{n} i^2 &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \\
&= \frac{n(n + \frac{1}{2})(n + 1)}{3}
\end{aligned}
$$

But be careful! This equation is valid only if we were correct in our initial guess at the form of the solution. If we were wrong, then this equation might not hold for any n other than 0, 1, 2, and 3! So be sure to verify that such guesses are correct by giving an induction proof.

3   Approximating Functions

When you ask about the weather, you probably want to hear something like "hot", "cold", or "raining", not a lengthly discourse on convection currents, radiative heat transfer, and dew points. Something similar is true about the analysis of computer algorithms and systems: a simple, approximate answer is often more useful than an exact, complicated answer. So techniques for replacing complicated expressions with simple approximations can be extremely useful.

3.1   Taylor's Theorem

A wonderful way to get approximations is to dredge up Taylor's Theorem from the dark, murky regions of calculus. Unfortuantely, the theorem *looks* like something dredged from a dark, murky place. So let's start with some intuition.

Suppose we want a simple approximation for the function $\ln(1 + x)$ that is fairly accurate when x is a small number. Here's what the function actually looks like:
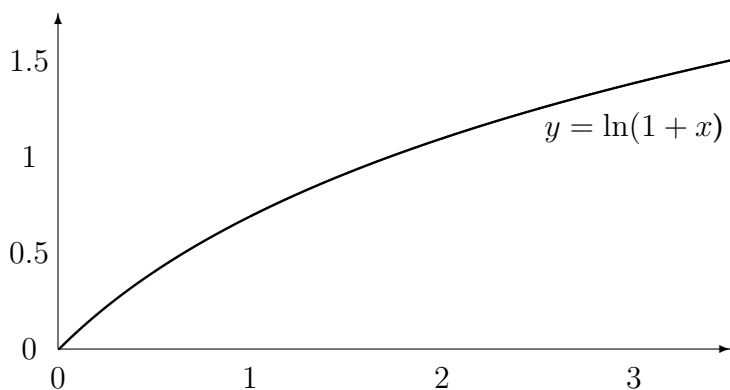
The curve goes through the origin, so a trivial approximation is $\ln(1 + x) \approx 0$ when x is small. In general:
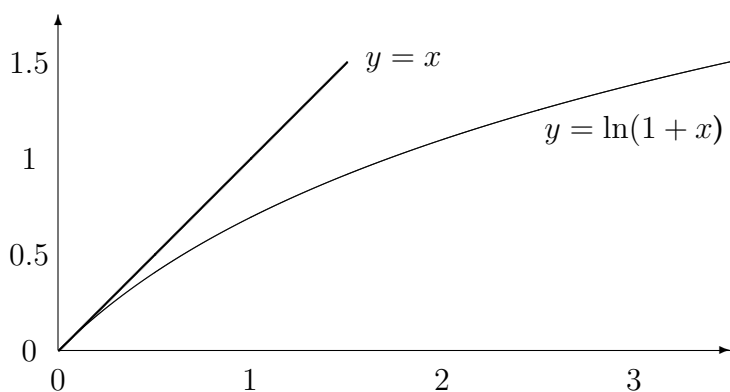
$$f(x) \approx f(0) \quad \text{when } x \text{ is small}$$

Sometimes this crude approximation is good enough, but not often.

A line tangent to the curve at $x = 0$ provides a better approximation. The slope of the tangent is given by the derivative

$$\frac{d}{dx} \ln(1 + x) = \frac{1}{1 + x}$$

evaluated at $x = 0$, which is $1/(1 + 0) = 1$. This gives the approximation $\ln(1 + x) \approx x$, which is a significant improvement:


In general:

$$f(x) \approx f(0) + xf'(0) \quad \text{when } x \text{ is small}$$

This approximation is accurate enough in many situations. This is great, because linear functions are very simple and easy to manipulate.

This linear approximation has both the correct value at $x = 0$ and the correct first derivative. We can go one more step and get an approximation with the correct second derivative:

$$f(x) \approx f(0) + xf'(0) + \frac{x^2}{2} f''(0) \quad \text{when } x \text{ is small}$$

In our example, where $f(x) = \ln(1 + x)$, we have $f''(x) = -1/(1 + x)^2$, which means $f''(0) = -1$. This gives the quadratic approximation $\ln(1 + x) \approx x - x^2/2$. Lets add this to our plot:
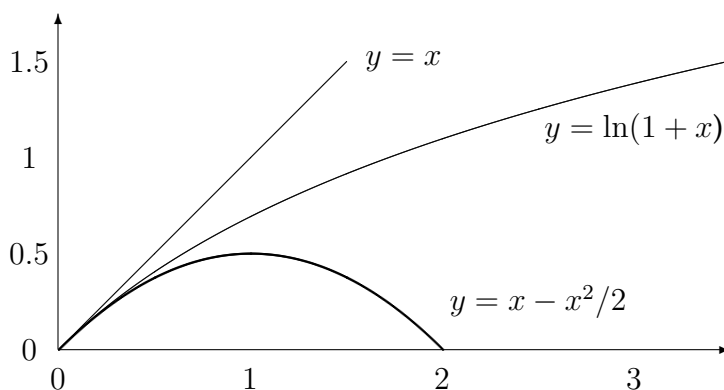
Taylor's Theorem says that one can continue in this way, getting progressively better approximations by incorporating higher-order derivatives.

Theorem (Taylor's theorem). *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is $(n + 1)$-times differentiable on the interval $[0, x]$. Then*

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \ldots + \frac{f^{(n)}(0)x^n}{n!} + \frac{f^{(n+1)}(z)x^{n+1}}{(n + 1)!}$$

for some $z \in [0, x]$.

The final term describes the "error" in an n-term Taylor approximation; we'll return to that later. Here are the leading terms in the Taylor series for some functions that often arise:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$$

$$\sqrt{1 + x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \ldots$$

3.2   Rule of 72

Suppose you want to double your money in n years. What annual rate of return would you need? Here are the answers for various numbers of years:

n = # of Years	Rate Needed to Double Money
4	18.9%
6	12.2%
8	9.0%
12	6.0%
24	2.9%
36	1.9%

There's a curious pattern here: the product of the numbers on each line is always around 70 or 72 or so.

   This observation is the basis for an old investing rule that says you can double your money by investing it for n years at $72/n$ percent interest. For example, you could double your money by investing it for 24 years at $72/24 = 3$ percent interest. This rule isn't exactly right, but it's almost mysteriously close. For example, a dollar invested for 24 years at 3% becomes $2.03.

   In general, a dollar invested for n years at $72/n$ percent interest becomes

$$m = \left(1 + \frac{72/n}{100}\right)^n$$

dollars. This complicated expression doesn't look much like the number 2, but let's try an approximation. One standard trick when the relevant variable (n in this case) appears in the exponent is to analyze the *logarithm* of the expression.

$$\ln m = \ln\left(1 + \frac{72/n}{100}\right)^n$$
$$= n \ln\left(1 + \frac{72}{100n}\right)$$

Here we're taking a logarithm of 1 plus something small ($72/100n$), so we can use the approximation $\ln(1 + x) \approx x - x^2/2$.

$$\ln m \approx n\left(\frac{72}{100n} - \frac{72^2}{2 \cdot 100^2 n^2}\right)$$
$$= \frac{72}{100} - \frac{162}{625n}$$

Now let's exponentiate to undo the earlier logarithm:

$$m \approx e^{72/100 - 162/(625n)}$$

This explains the Rule of 72: if you invest a dollar for n years at $72/n$ percent interest, then the value of your dollar approaches $e^{72/100} = 2.05443$ dollars from below as n goes to infinity. So the Rule overestimates your return for small n, slightly underestimates for large n, and is fairly accurate for typical values of n. A "Rule for 71" might be even more accurate, but 72 has lots of convenient divisors, while 71 is inconveniently prime.

3.3   Upper and Lower Bounds

Sometime an upper or lower bound on a function is needed instead of a simple approximation. For example, if you're computing the probability that a critical system fails, an upper bound would give you a reassuring, conservative answer. We can get good upper and lower bounds on many functions by looking at the error term in Taylor's Theorem.

Taylor's Theorem with one ordinary term and the error term says:

$$f(x) = f(0) + xf(z) \quad \text{for some } z \in [0, x]$$

This is actually an equation, not an approximation. The problem is that we don't know the value of z in the error term. However, we do have some bounds on z, and those bounds imply upper and lower bounds on the function $f(x)$.
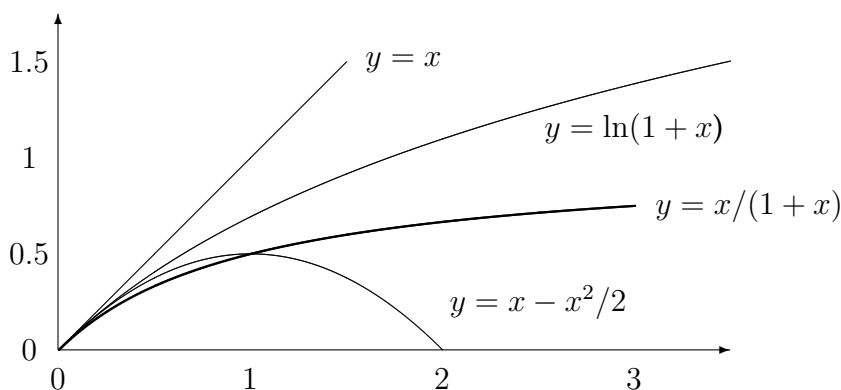
For example, suppose $f(x) = \ln(1 + x)$. Then Taylor's Theorem says:

$$\ln(1 + x) = 0 + x \cdot \frac{1}{1 + z} = \frac{x}{1 + z}$$

for some $z \in [0, x]$. If we assume x is positive, then the expression on the right is *maximized* when $z = 0$ and *minimized* when $z = x$. Therefore, we have the bounds:

$$\frac{x}{1 + x} \le \ln(1 + x) \le x$$

The upper bound is the same as the linear Taylor approximation. Let's add the lower bound to our plot:

This lower bound is fairly accurate for small x. However, for large x, the lower bound $x/(1 + x)$ asymptotically approaches 1 while the actual function $\ln(1 + x)$ diverges to infinity.

3.4 Approximating $1 + x$

We can get upper and lower bounds on the function $1 + x$ by exponentiating our bounds on $\ln(1 + x)$:

$$e^{x/(1+x)} \leq 1 + x \leq e^x$$

What a profoundly stupid thing to do! The function $1 + x$ is *really simple*, while the function $e^{x/(1+x)}$ is *complicated*. Why would one ever want to bound a simple function with a complicated function? Why not just work with the original, simple function?

The reason is that x appears at "ground level" in the function $1 + x$, but "upstairs" in the functions e^x and $e^{x/(1+x)}$. This difference allows one to switch back and forth between sums and products. As a result, $1 + x \approx e^x$ is probably the most widely-used approximation in computer science!

For example, let's take the nasty product that we tried to scare you with earlier:

$$\left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2}{n^2}\right)\left(1 + \frac{3}{n^2}\right)\cdots\left(1 + \frac{n}{n^2}\right)$$

Our upper bound on $1 + x$ gives:

$$\left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2}{n^2}\right)\cdots\left(1 + \frac{n}{n^2}\right) \leq e^{1/n^2} \cdot e^{2/n^2} \cdots e^{n/n^2}$$
$$= e^{(1+2+\ldots+n)/n^2}$$
$$= e^{n(n+1)/2n^2}$$
$$= \sqrt{e} \cdot e^{1/2n}$$

The second line follows from the rule $e^A \cdot e^B = e^{A+B}$. We've now translated the nasty product into an easy sum in the exponent! The rest is simplification. Note that $e^{1/2n}$ is very close to 1 when n is large.

Similarly, the lower bound on $1 + x$ gives:

$$\left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2}{n^2}\right)\cdots\left(1 + \frac{n}{n^2}\right) \geq e^{\frac{1/n^2}{1+1/n^2}} \cdot e^{\frac{2/n^2}{1+2/n^2}} \cdots e^{\frac{n/n^2}{1+n/n^2}}$$
$$= e^{\frac{1}{n^2+1} + \frac{2}{n^2+2} + \ldots + \frac{n}{n^2+n}}$$

This sum is outside our repetoire. But we can get a lower bound by making some of the denominators a little bigger. In particular, if we make all the denominators as big as the largest one, then we have:

$$\left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2}{n^2}\right)\cdots\left(1 + \frac{n}{n^2}\right) \geq e^{\frac{1}{n^2+n} + \frac{2}{n^2+n} + \ldots + \frac{n}{n^2+n}}$$
$$= e^{(1+2+\ldots+n)/n(n+1)}$$
$$= \sqrt{e}$$

So the very scary product rapidly descends to nothing more than $\sqrt{e}$.

3.5 A Large-x Approximation

So far, we've focused on the problem of approximating a function $f(x)$ when x is small. But what if we need to approximate a function $f(x)$ when x is *large*?

Here's a simple trick that often works. As an example, let's take our old friend $f(x) = \ln(1 + x)$. When x is a big number, there is little difference between $\ln x$ and $\ln(1 + x)$. So let's first write $\ln(1 + x)$ in terms of $\ln x$:

$$\ln(1 + x) = \ln\left(x \cdot \left(1 + \frac{1}{x}\right)\right)$$
$$= \ln x + \ln\left(1 + \frac{1}{x}\right)$$

The general idea is to express $f(x)$ in terms of a fairly similar function and then use Taylor's Theorem just to analyze the leftover bit, which is $\ln(1 + 1/x)$ in this case. Notice that when x is large, $1/x$ is small. So our earlier methods should now give excellent results. Specifically, suppose we set $z = 1/x$ and use the bounds that we derived earlier:

$$\frac{z}{1 + z} \leq \ln(1 + z) \leq z$$

Then we get the upper bound

$$\ln(1 + x) \leq \ln x + \frac{1}{x}$$

and the lower bound:

$$\ln(1 + x) \geq \ln x + \frac{1/x}{1 + 1/x}$$
$$\geq \ln x + \frac{1}{x + 1}$$

For example, suppose we wanted to evaluate $\ln(e^{10} + 1)$. Then these bounds say:

$$10 + \frac{1}{e^{10} + 1} \leq \ln(e^{10} + 1) \leq 10 + \frac{1}{e^{10}}$$

The difference between the upper and lower bounds is only about e^{-20}. Not bad!