

Quiz 1

YOUR NAME: _____

Circle the name of your recitation instructor:

Ishan Christos Grant

- You may use one 8.5×11 " sheet with notes in you own handwriting on both sides, but no other sources of information.
- Calculators are not allowed.
- You may assume all results from lecture, the notes, problem sets, and recitation.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- The exam ends at 9:30 PM.
- GOOD LUCK!

Problem	Points	Grade	Grader
1	20		
2	15		
3	20		
4	15		
5	15		
6	15		
Total	100		

Problem 1. [20 points]

(a) Consider the proposition:

$$R = \text{"For all } x \in S, P(x) \text{ implies } Q(x)."$$

For each statement below:

- If R *implies* that statement, then circle $\Rightarrow$.
- If R is *implied by* that statement, then circle $\Leftarrow$.

Thus, you might circle zero, one, or two arrows next to each statement. (Circle only implications that hold for *all* sets S and *all* predicates P and Q .)

$\Rightarrow$ $\Leftarrow$ For all $x \in S$, $Q(x)$ implies $P(x)$.

$\Rightarrow$ $\Leftarrow$ For all $x \in S$, $\neg Q(x)$ implies $\neg P(x)$.

$\Rightarrow$ $\Leftarrow$ For all $x \in S$, $P(x)$ and $Q(x)$.

$\Rightarrow$ $\Leftarrow$ There does not exist an $x \in S$ such that not ($P(x)$ implies $Q(x)$).

$\Rightarrow$ $\Leftarrow$ Pigs fly.

- (b) Let S be the set of all people, and let $M(x, y)$ be the predicate, “ x is the mother of y ”. Translate this proposition into a clear English sentence involving no variables.

$$\forall x (\neg \exists y (M(x, y) \wedge M(y, x)))$$

“There are no two people such that each is the mother of the other.” Or, more simply, “No one is their own maternal grandmother.”

- (c) Translate the following English sentence into logic notation using the set S and predicate M defined above.

“Everyone has a mother.”

$$\forall x \exists y M(y, x)$$

Problem 2. [15 points] Complete this proof that n cents of postage can be formed using 3 and 5 cent stamps for all $n \geq 8$.

Proof. We use strong induction.

(a) Let $P(n)$ be the proposition that

Solution. n cents of postage can be formed using 3 and 5 cent stamps.

(b) *Base cases.*

Solution. $P(8)$, $P(9)$, and $P(10)$ are all true, since:

$$8 = 5 + 3$$

$$9 = 3 + 3 + 3$$

$$10 = 5 + 5$$

(c) *Inductive step.*

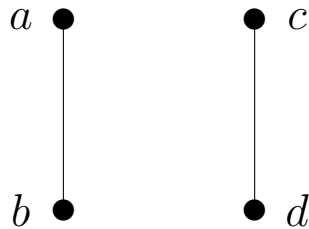
Solution. For $n \geq 10$, we assume $P(8), \dots, P(n)$ and prove $P(n + 1)$. In particular, by assumption $P(n - 2)$, we can form $n - 2$ cents of postage. Adding a 3-cent stamp gives $n + 1$ cents of postage, so $P(n + 1)$ is true.

So $P(n)$ is true for all $n \geq 8$ by the principle of strong induction.

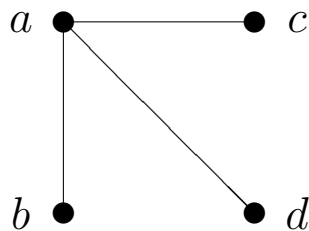
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Problem 3. [20 points] Here is how to *tweak* an undirected graph:

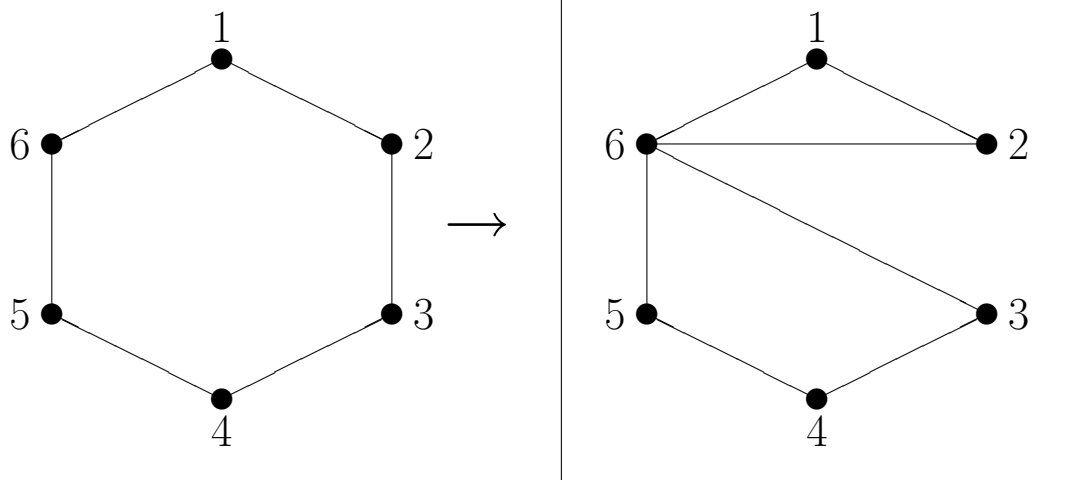
1. Select distinct vertices a, b, c , and d such that the graph contains edges $a-b$ and $c-d$ and none of the edges $a-c$, $a-d$, $b-c$, or $b-d$.



2. Delete edge $c-d$ and add edges $a-c$ and $a-d$:



- (a) In the box on the right, draw a graph that can be obtained by tweaking the graph on the left.



(b) Suppose that G_0 is an undirected graph with an Euler tour. Also, suppose G_1 is obtained by tweaking G_0 , G_2 by tweaking G_1 , and so forth. Use induction to prove that every graph G_n obtainable in this way has an Euler tour.

For your reference:

- An **Euler tour** is a closed walk that traverses every edge in a graph exactly once.
- A graph is **connected** if and only if there is a path between every pair of vertices.
- **Theorem.** An undirected graph has an Euler tour if and only if the graph is connected and every vertex has even degree.

Solution. We use induction. Let $P(n)$ be the proposition that G_n has an Euler tour.

Base case. G_0 has an Euler tour by supposition.

Inductive step. For $n \geq 0$, we assume G_n has an Euler Tour and prove that G_{n+1} also has an Euler tour. Specifically, we show that G_{n+1} has only even-degree vertices and is connected:

- Every vertex in G_n has even degree, since G_n has an Euler tour. Every vertex in G_{n+1} has the same degree, except for vertex a which has degree two greater. Thus, every vertex in G_{n+1} has even degree.
- Consider arbitrary vertices u and v in G_{n+1} . Since G_n is connected, there is a path from u to v in G_n . If the path does not contain $c-d$, then the same path exists in G_{n+1} . If the path does contain $c-d$, then there is a corresponding path in G_{n+1} where $c-d$ is replaced by edges $c-a$ and $a-d$.

This implies G_{n+1} has an Euler tour as well. Therefore, G_n has an Euler tour for all $n \geq 1$. In particular, G_{6042} has an Euler Tour.

Problem 4. [15 points] Fill in the boxes below. All variables denote integers. No explanations are required, but we can only award partial credit for an incorrect answer if you show your reasoning.

- (a) Suppose x is a multiple of 17. Write the smallest **nonnegative** integers that make this statement true.

$$2x^{32} - 6x^{17} + 4x^{16} - 4x + 6 \equiv \boxed{0} \cdot x + \boxed{6} \pmod{17}$$

Solution. If x is a multiple of 17, then $x \equiv 0 \pmod{17}$. Therefore, all terms involving x on the left are congruent to zero.

- (b) Now suppose x is *not* a multiple of 17. Write the smallest **nonnegative** integers that make this statement true.

$$2x^{32} - 6x^{17} + 4x^{16} - 4x + 6 \equiv \boxed{15} \cdot x + \boxed{12} \pmod{17}$$

Solution. By Fermat's Theorem, $x^{16} \equiv 1 \pmod{17}$. Thus, we can reason as follows:

$$\begin{aligned} 2x^{32} - 6x^{17} + 4x^{16} - 4x + 6 &\equiv 2(x^{16})^2 - 6x(x^{16}) + 4x^{16} - 4x + 6 \pmod{17} \\ &\equiv 2 - 6x + 4 - 4x + 6 \pmod{17} \\ &\equiv -2x + 12 \pmod{17} \\ &\equiv 15x + 12 \pmod{17} \end{aligned}$$

- (c) In the box, write the smallest **positive** integer that makes this statement true:

There exist integers s and t such that

$$s \cdot 117 + t \cdot 153 = x$$

if and only if

$$x \equiv 0 \pmod{\boxed{9}}$$

Solution. Recall that an integer x is expressible as a linear combination of a and b if and only if x is a multiple of $\gcd(a, b)$, i.e. $x \equiv 0 \pmod{\gcd(a, b)}$. In this case, Euclid's algorithm gives:

$$\gcd(153, 117) = \gcd(117, 36) = \gcd(36, 9) = 9$$

Problem 5. [15 points] Let p , q , and r be distinct primes. Prove that there exist integers a , b , and c such that:

$$a \cdot (pq) + b \cdot (qr) + c \cdot (rp) = 1$$

(Hint: First, consider linear combinations of just pq and qr .)

Solution. Since $\gcd(pq, qr) = q$, there exist integers s and t such that:

$$s(pq) + t(qr) = q$$

Now $\gcd(q, rp) = 1$, so there exist integers u and v such that:

$$uq + v(rp) = 1$$

Therefore:

$$u(s(pq) + t(qr)) + v(rp) = (us)(pq) + (ut)(qr) + v(rp) = 1$$

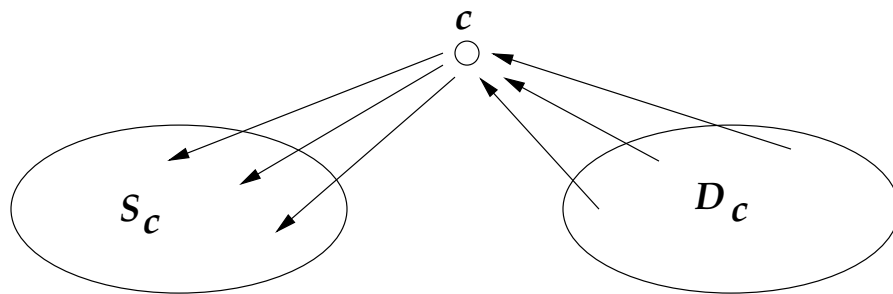
Problem 6. [15 points] In a chicken tournament, for every pair of chickens u and v , either u pecks v or v pecks u , but not both. A *king* is a chicken u such that for every other chicken v , either

- u pecks v , or
- u pecks a chicken w and w pecks v .

Complete the proof of the following theorem.

Theorem. *If chicken c is pecked, then c is pecked by a king.*

Proof. Let S_c be the set of all chickens pecked by c , and let D_c be the set of all chickens that peck c . The situation is illustrated below:



(Hint: Apply the King Chicken Theorem to D_c .)

If chicken c is pecked, then the set D_c is nonempty. Thus, there is a tournament among the chickens in D_c , which has a king by the King Chicken Theorem. We will show that d is actually a king of the original tournament.

- d pecks every chicken in D_c (directly or indirectly), since it is a king of D_c .
- d pecks chicken c directly, since d is in D_c .
- d pecks every chicken in S_c indirectly, since it pecks c and c pecks every chicken in S_c .

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