Srini Devadas and Eric Lehman

Quiz 1

YOUR NAME: ____________________________

Circle the name of your recitation instructor:

Ishan Christos Grant

- You may use one $8.5 \times 11''$ sheet with notes in you own handwriting on both sides, but no other sources of information.
- Calculators are not allowed.
- You may assume all results from lecture, the notes, problem sets, and recitation.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- The exam ends at 9:30 PM.
- GOOD LUCK!

Problem	Points	Grade	Grader
1	20		
2	15		
3	20		
4	15		
5	15		
6	15		
Total	100		

Problem 1. [20 points]

(a) Consider the proposition:

$$R = \text{“For all } x \in S,\ P(x) \text{ implies } Q(x).\text{”}$$

For each statement below:

- If R *implies* that statement, then circle $\Rightarrow$.
- If R is *implied by* that statement, then circle $\Leftarrow$.

Thus, you might circle zero, one, or two arrows next to each statement. (Circle only implications that hold for *all* sets S and *all* predicates P and Q.)

$\Rightarrow$	$\Leftarrow$	For all $x \in S$, $Q(x)$ implies $P(x)$.
$\boxed{\Rightarrow}$	$\boxed{\Leftarrow}$	For all $x \in S$, $\neg Q(x)$ implies $\neg P(x)$.
$\Rightarrow$	$\boxed{\Leftarrow}$	For all $x \in S$, $P(x)$ and $Q(x)$.
$\boxed{\Rightarrow}$	$\boxed{\Leftarrow}$	There does not exist an $x \in S$ such that not ($P(x)$ implies $Q(x)$).
$\Rightarrow$	$\boxed{\Leftarrow}$	Pigs fly.

(b) Let S be the set of all people, and let $M(x, y)$ be the predicate, "x is the mother of y". Translate this proposition into a clear English sentence involving no variables.

$$\forall x \; (\neg \exists y \; (M(x, y) \wedge M(y, x)))$$

"There are no two people such that each is the mother of the other." Or, more simply, "No one is their own maternal grandmother."

(c) Translate the following English sentence into logic notation using the set S and predicate M defined above.

"Everyone has a mother."

$$\forall y \; \exists x \; M(x, y)$$

Problem 2. [15 points] Complete this proof that n cents of postage can be formed using 3 and 5 cent stamps for all $n \geq 8$.

Proof. We use strong induction.

(a) Let $P(n)$ be the proposition that

Solution. n cents of postage can be formed using 3 and 5 cent stamps.

(b) *Base cases.*

Solution. $P(8)$, $P(9)$, and $P(10)$ are all true, since:

$$\begin{aligned} 8 &= 5 + 3 \\ 9 &= 3 + 3 + 3 \\ 10 &= 5 + 5 \end{aligned}$$

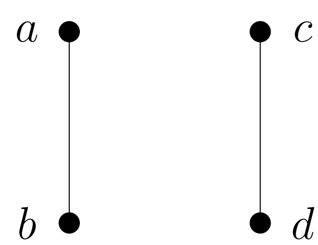
(c) *Inductive step.*

Solution. For $n \geq 10$, we assume $P(8), \ldots, P(n)$ and prove $P(n+1)$. In particular, by assumption $P(n-2)$, we can form $n-2$ cents of postage. Adding a 3-cent stamp gives $n+1$ cents of postage, so $P(n+1)$ is true.

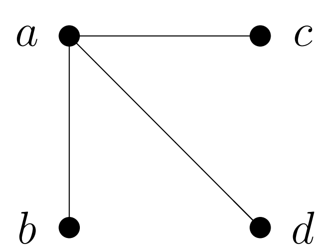
So $P(n)$ is true for all $n \geq 8$ by the principle of strong induction. □

Problem 3. [20 points] Here is how to ***tweak*** an undirected graph:

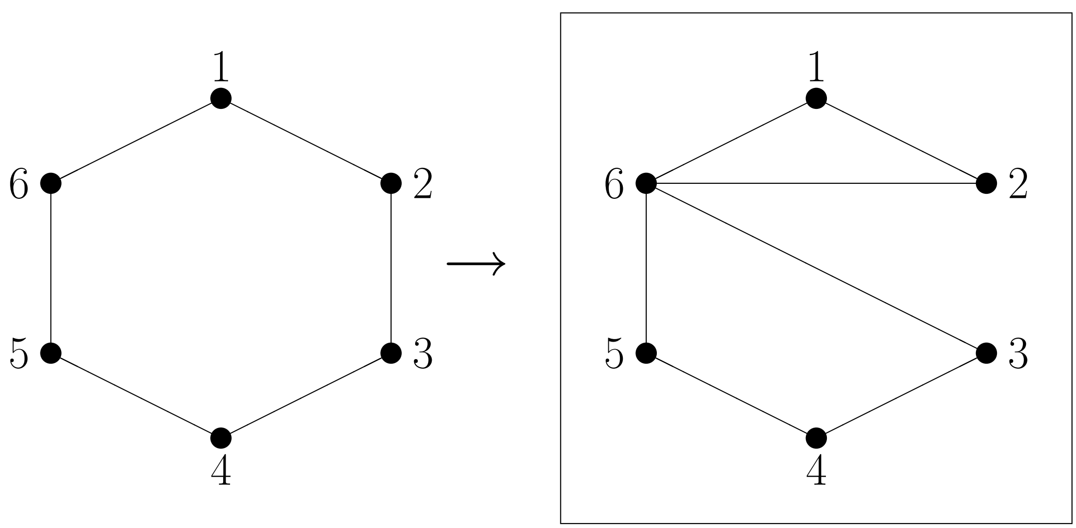
1. Select distinct vertices a, b, c, and d such that the graph contains edges a—b and c—d and none of the edges a—c, a—d, b—c, or b—d.



2. Delete edge c—d and add edges a—c and a—d:



(a) In the box on the right, draw a graph that can be obtained by tweaking the graph on the left.



(b) Suppose that G_0 is an undirected graph with an Euler tour. Also, suppose G_1 is obtained by tweaking G_0, G_2 by tweaking G_1, and so forth. Use induction to prove that every graph G_n obtainable in this way has an Euler tour.

For your reference:

- An ***Euler tour*** is a closed walk that traverses every edge in a graph exactly once.
- A graph is ***connected*** if and only if there is a path between every pair of vertices.
- **Theorem.** An undirected graph has an Euler tour if and only if the graph is connected and every vertex has even degree.

Solution. We use induction. Let $P(n)$ be the proposition that G_n has an Euler tour.

Base case. G_0 has an Euler tour by supposition.

Inductive step. For $n \geq 0$, we assume G_n has an Euler Tour and prove that G_{n+1} also has an Euler tour. Specifically, we show that G_{n+1} has only even-degree vertices and is connected:

- Every vertex in G_n has even degree, since G_n has an Euler tour. Every vertex in G_{n+1} has the same degree, except for vertex a which has degree two greater. Thus, every vertex in G_{n+1} also has even degree.
- Consider arbitrary vertices u and v. Since G_n is connected, there is a path from u to v in G_n. If the path does not contain c—d, then the same path exists in G_{n+1}. If the path does contain c—d, then there is a corresponding path in G_{n+1} where c—d is replaced by edges c—a and a—d. Thus, there is a path between every pair of vertices in G_{n+1}, which means that G_{n+1} is connected.

These two facts imply G_{n+1} has an Euler tour. By induction, every graph G_n obtainable by repeatedly tweaking G_0 has an Euler tour.

Problem 4. [15 points] Fill in the boxes below. All variables denote integers. No explanations are required, but we can only award partial credit for an incorrect answer if you show your reasoning.

(a) Suppose x is a multiple of 17. Write the smallest **nonnegative** integers that make this statement true.

$$2x^{32} - 6x^{17} + 4x^{16} - 4x + 6 \equiv \boxed{0} \cdot x + \boxed{6} \pmod{17}$$

Solution. If x is a multiple of 17, then $x \equiv 0 \pmod{17}$. Therefore, all terms involving x on the left are congruent to zero.

(b) Now suppose x is *not* a multiple of 17. Write the smallest **nonnegative** integers that make this statement true.

$$2x^{32} - 6x^{17} + 4x^{16} - 4x + 6 \equiv \boxed{7} \cdot x + \boxed{12} \pmod{17}$$

Solution. By Fermat's Theorem, $x^{16} \equiv 1 \pmod{17}$. Thus, we can reason as follows:

$$\begin{aligned} 2x^{32} - 6x^{17} + 4x^{16} - 4x + 6 &\equiv 2\left(x^{16}\right)^2 - 6x\left(x^{16}\right) + 4x^{16} - 4x + 6 \pmod{17} \\ &\equiv 2 - 6x + 4 - 4x + 6 \pmod{17} \\ &\equiv -10x + 12 \pmod{17} \\ &\equiv 7x + 12 \pmod{17} \end{aligned}$$

(c) In the box, write the smallest **positive** integer that makes this statement true:

There exist integers s and t such that

$$s \cdot 117 + t \cdot 153 = x$$

if and only if

$$x \equiv 0 \pmod{\boxed{9}}$$

Solution. Recall that an integer x is expressible as a linear combination of a and b if and only if x is a multiple of $\gcd(a, b)$, i.e. $x \equiv 0 \pmod{\gcd(a, b)}$. In this case, Euclid's algorithm gives:

$$\gcd(153, 117) = \gcd(117, 36) = \gcd(36, 9) = 9$$

Problem 5. [15 points] Let p, q, and r be distinct primes. Prove that there exist integers a, b, and c such that:

$$a \cdot (pq) + b \cdot (qr) + c \cdot (rp) = 1$$

(Hint: First, consider linear combinations of just pq and qr.)

Solution. Since $\gcd(pq, qr) = q$, there exist integers s and t such that:

$$s(pq) + t(qr) = q$$

Now $\gcd(q, rp) = 1$, so there exist integers u and v such that:

$$uq + v(rp) = 1$$

Therefore:

$$u(s(pq) + t(qr)) + v(rp) = (us)(pq) + (ut)(qr) + v(rp) = 1$$

Let $a = us$, $b = ut$, and $c = v$.

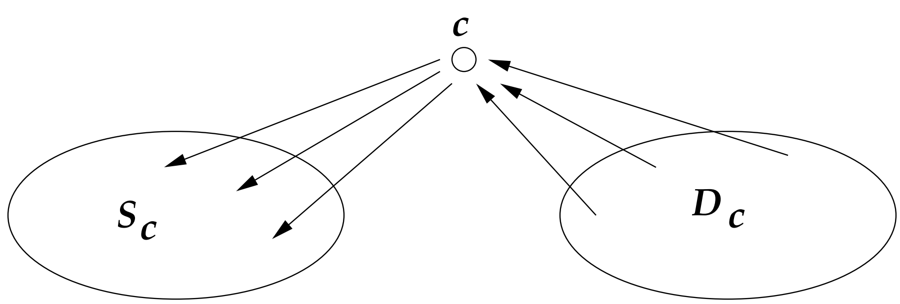
Problem 6. [15 points] In a chicken tournament, for every pair of chickens u and v, either u pecks v or v pecks u, but not both. A *king* is a chicken u such that for every other chicken v, either

- u pecks v, or
- u pecks a chicken w and w pecks v.

Complete the proof of the following theorem.

Theorem. *If chicken c is pecked, then c is pecked by a king.*

Proof. Let S_c be the set of all chickens pecked by c, and let D_c be the set of all chickens that peck c. The situation is illustrated below:



(Hint: Apply the King Chicken Theorem to D_c.)

If chicken c is pecked, then the set D_c is nonempty. Thus, there is a mini-tournament among the chickens in D_c, which has a king d by the King Chicken Theorem. We will show that d is actually a king of the entire tournament.

- d pecks every chicken in D_c (directly or indirectly), since it is a king of D_c.
- d pecks chicken c directly, since d is in D_c.
- d pecks every chicken in S_c indirectly, since it pecks c and c pecks every chicken in S_c.

Thus, c is pecked by a king; namely, d. □