Solutions to Quiz 2

Problem 1 (20 points). Structural Induction

The set of binary strings that match up (*BMU*'s) is defined inductively as follows:

- The empty string, λ, is in BMU. (BMU-base)
- If $s, t \in$ BMU, then $0s1t \in$ BMU. (BMU-ind)

In this problem we ask you to complete a proof by structural induction that BMU is closed under concatenation, that is

If $s, t \in$ BMU, then the string st is in BMU.

The induction hypothesis will be

$$P(r) ::= \quad \forall t' \in \text{BMU}\, [rt' \in \text{BMU}].$$

(a) (5 points) Prove the base case of the structural induction.

Solution. The base case is $P(\lambda)$, that is,

$$\forall t' \in \text{BMU}\, [\lambda t' \in \text{BMU}].$$

Since λ is the empty string, $\lambda t'$ is just t'. Hence, $P(\lambda)$ is really

$$\forall t' \in \text{BMU}\, [t' \in \text{BMU}],$$

which is trivially true. ■

(b) (15 points) Below is a proof of the inductive step of the structural induction, that is, a proof that if $r = 0s1t$ for $s, t \in$ BMU, then $P(r)$ holds. Some possible justifications for lines of the proof are listed on the next page. Fill in the number of the justification of the various lines of the proof where indicated.

Proof. Suppose t' is some string in BMU. According to _____, we must prove that $rt' \in$ BMU.

Solution. *The correct answer was "(8): definition of $P(r)$" and was worth 3 points. Partial credit of 2 points was given to answer "(4): structural induction hypothesis for r" —a subtle difference, really.* ■

Then

$$tt' \in \text{BMU} \qquad \text{(by \_\_\_\_\_).}$$

Solution. *(1): structural induction hypothesis for t [3 points].* ■

Let $t'' ::= tt'$. Now

$$rt' = (\texttt{0}s\texttt{1}t)t' \qquad \text{(by \_\_\_\_\_)}$$

Solution. *(5): definition of r [2 points].* ■

$$= \texttt{0}s\texttt{1}(tt') \qquad \text{(by \_\_\_\_\_)}$$

Solution. *(10): associativity of string concatenation [2 points].* ■

$$= \texttt{0}s\texttt{1}t'' \qquad \text{(by \_\_\_\_\_)}$$

Solution. *(7): definition of t'' [2 points].* ■

Therefore, $rt' \in$ BMU by _____

Solution. *(14) case (BMU-ind) of the definition of BMU [3 points].* ■

□

Justifications:

1. structural induction hypothesis for t.
2. structural induction hypothesis for t'.
3. structural induction hypothesis for s.
4. structural induction hypothesis for r.
5. definition of r.
6. definition of t'.
7. definition of t''.
8. definition of $P(r)$.

9. definition of $P(t')$.
10. associativity of string concatenation.
11. $\lambda \in$ BMU.
12. BMU $=$ strings that "count right".
13. Case (BMU-base) of the definition of BMU.
14. Case (BMU-ind) of the definition of BMU.

Problem 2 (20 points). A deck of cards numbered $1, \ldots, 2n$, where n is a positive integer, is shuffled and placed on a table. The following actions are carried out:

(*) A card that is not at the bottom of the deck is picked. Then,

1. if the card below the one picked has a lower number, then the positions in the deck of the two cards are switched, and step (*) is repeated.
2. if the card below the one picked has a higher number, then both cards are removed from the deck and discarded. If there are still cards in the deck, *the deck is reshuffled* and step (*) is repeated. Otherwise, the process is over.

This process can be modelled as a state machine whose states describe the situation when a card is about to be picked at step (*). Namely, a state will be the sequence S, of (the numbers of) the cards in the deck on the table, from top to bottom.

(a) (3 points) What are the possible start states of the machine?

Solution. Any sequence of the integers $1, \ldots, 2n$.

Grading Policy: You received full credit if you conveyed that the start states were all possible permutations of the sequence of integers $1, \ldots, 2n$. You received no credit if you did not correctly identify the start states.

You received 1 point if instead of a sequence of numbers, you used the language "cards," "deck," *etc.*, to represent a state. Remember that a State Machine has no notion of what a "card" or "deck" is. It is necessary to *represent* of the idea behind a "deck" with a standard mathematical object such as a sequence of numbers.

If you have any questions, or feel that your grade does not reflect this grading policy, please contact TA Paul Youn at `youn@mit.edu`. ■

(b) (5 points) Transitions $S \longrightarrow S'$ corresponding to case 1 can be specified by the conditions that there are numbers $k > l$ such that $S = S_1 k l S_2$, and $S' = S_1 l k S_2$.

Describe the conditions on S and S' that specify a transition $S \longrightarrow S'$ corresponding to case 2.

Solution. There are $k < l$ such that $S = S_1 k l S_2$, and S' is a sequence of the same numbers as are in the sequence $S_1 S_2$.

Common Mistakes: Some students simply wrote that $S' = S_1 S_2$, which ignores the reshuffling of the deck following case 2. This reshuffling aspect is the main part of the problem. Indeed, without reshuffling the deck, the solution would be almost identical to the example given.

Grading Policy: Failure to somehow capture the idea that the deck was reshuffled was -3 points. Imprecise language was -1 point.

Writing something like: $|S'| = |S_1 S_2|$ was -1 point. This indicated the right idea, and is a necessary condition, does not really describe the reshuffling. You needed to point out that not only are the sets the same size, but also that the same cards are in S' as are in $S_1 S_2$. ■

(c) (12 points) Indicate next to each of the following derived variables which of these properties it has:

constant	C
strictly increasing	SI
strictly decreasing	SD
weakly increasing but not constant	WI
weakly decreasing but not constant	WD
none of the above	N

- $|S|$ ::= the size of the deck _____

  Solution. Weakly Decreasing.

  The first transition maintains the size of the deck, while the second transition decreases the size of the deck.

  Grading Policy: Each part was graded all or none (2 points per part). ■

- α::= the number of "out-of-order" pairs (i, j) such that $i > j$ but card i is above card j in the deck. _____

  Solution. None of the Above.

  The first transition decreases the number of "out-of-order" pairs (it removes an out of order pair and otherwise leaves the deck as it was). The second transition, however, can increase *or* decrease the number of out-of-order pairs. If the deck originally has 10 out-of-order pairs, after the reshuffling it can have less or more out-of-order pairs.

  Common Mistakes: Some people thought this was a decreasing quantity because at the end of the run, the deck is empty and has no out-of-order pairs left. ■

- $|S| + \alpha$ _____

  Solution. None of the Above.

  While S is weakly decreasing, α is willy nilly all over the place (that phrase should be used more often). So, for example, in case 2, after 2 cards are removed $|S|$ decreases by 2, but who knows how many out of order pairs exist after the reshuffling. ■

- $(\alpha, |S|)$ under lexicographic order _____

  Solution. None of the Above.

  Under the first transition, α decreases while $|S|$ stays the same. Right there we know that the quantity must be some decreasing form or none of the above. Under the second transition, however, we don't know what happens to α. It could increase, or decrease, or stay the same. ■

- $(|S|, \alpha)$ under lexicographic order _____

  Solution. Strictly Decreasing.

  In the first transition, the size of the deck stays the same, while α decreases by 1. In the second transition, $|S|$ decreases by 2 as 2 cards are removed, and α does something. However, since the first term in the pair decreases, it doesn't matter what the second term does under lexigraphical ordering.

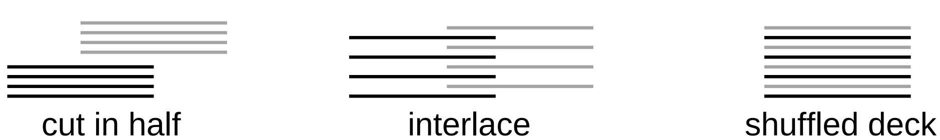
  ■

- $n^2 |S| + \alpha$ _____

  Solution. Strictly Decreasing.

  This one was kind of tough. In the first transition, α decreases and $|S|$ stays the same. The quantity decreases. In the second transition, the $n^2 |S|$ decreases by $2n^2$, while α does something. How much can α increase by? The worst case is that the deck has no out-of-order pairs, and then the reshuffling puts the entire deck in reverse order (maximum out-of-order pairs). In this case, the top card is out of order with the $2n - 1$ cards below, the next card is out of order with the $2n - 2$ cards below, *etc*. This yields the familiar closed form $2n(2n-1)/2 = 2n^2 - n$ which is less than $2n^2$. So, the quantity again decreases. ■

Problem 3 (20 points). We have a deck with 54 cards. (We left the two jokers in.) In a *perfect shuffle*, we cut the deck exactly in half and then interlace the two halves. The top card remains on the top after the shuffle, and the bottom card remains on the bottom. A perfect shuffle with an 8-card deck is illustrated below:



Suppose there are initially i cards above card C. Then number of cards above C after a perfect shuffle is congruent to $2i$ modulo 53. You may assume this without proof, though it is not difficult to verify.

(a) (5 points) Suppose there are initially i cards above card C. The number of cards above C after k perfect shuffles must be congruent to ____________ modulo 53.

(b) (15 points) Prove that 52 successive perfect shuffles restore the deck to its original configuration.

Solution. The top and bottom cards never move. Consider a card C with $i \in \{1, 2, \ldots, 52\}$ cards above. After 52 perfect shuffles:

$$\begin{aligned}
\text{\# of cards above } C &\equiv 2^{52} \cdot i \pmod{53} && \text{(by part (a))} \\
&\equiv 1 \cdot i \pmod{53} && \text{(by Fermat's Theorem)} \\
&\equiv i \pmod{53}
\end{aligned}$$

If the number of cards above C is congruent to $i \in \{1, 2, \ldots, 52\}$, then it must be equal to i. Thus, every card returns to its original position and the deck is restored after 52 perfect shuffles. ■

Problem 4 (20 points). Sammy the Shark is a financial service provider who offers loans on the following terms.

- Sammy loans a client m dollars in the morning. This puts the client m dollars in debt to Sammy.
- Each evening, Sammy first charges a "service fee", which increases the client's debt by f dollars, and then Sammy charges interest, which multiplies the debt by a factor of p. For example, if Sammy's interest rate was 5% per day, then p would be 1.05.

(a) (3 points) What is the client's debt at the end of the first day?

Solution. At the end of the first day, the client owes Sammy $(m + f)p = mp + fp$ dollars.

Grading Policy: There was one point taken off if 1.05 was used instead of p, or $1 + p$ was used instead of p. ■

(b) (5 points) What is the client's debt at the end of the second day?

Solution. $((m+f)p+f)p = mp^2 + fp^2 + fp$

Grading Policy: Almost everybody got either full credit, or no credit on this. One point was taken off if the second day was confused for the first day. ■

(c) (12 points) Write a formula for the client's debt after d days. A correct *closed form* formula gets full credit; a correct formula *not* in closed form gets half credit.

Solution. The client's debt after three days is

$$(((m+f)p+f)p+f)p = mp^3 + fp^3 + fp^2 + fp.$$

Generalizing from this pattern, the client owes

$$mp^d + \sum_{k=1}^{d} fp^k \tag{1}$$

dollars after d days. Applying the formula for a geometric sum to (1) gives:

$$mp^d + f \cdot \left(\frac{p^{d+1}-1}{p-1} - 1 \right).$$

Grading Policy: Correct, closed form solutions were given 12 points. Correct formulas not in closed form were given 6 points. In addition, partial credit was given if incorrect closed form solutions were derived from correct formulas not in closed form. ■

Problem 5 (20 points). Prove that $2\ln(n^3)+2$ and $\sum_{i=1}^{n} 6/i$ are asymptotically equal ($\sim$). You may assume any results from class, text, or Notes in your proof.

Solution. In Course Notes 10, we have shown that $H_n \sim \ln n$. Therefore,

$$\sum_{i=1}^{n} 6/i = 6H_n \sim 6\ln n \sim 6\ln n + 2 = 2\ln(n^3) + 2.$$

We can prove $6\ln n \sim 6\ln n + 2$ by

$$\lim_{n\to\infty} \frac{6\ln n + 2}{6\ln n} = \lim_{n\to\infty} \left(1 + \frac{2}{6\ln n}\right) = 1$$

Grading Policy:

-5 points for claiming

$$\frac{\infty}{\infty} = 1$$

or not deriving the limit correctly.

-5 points for not citing or proving $H_n \sim \ln n$.

■