Notes for Tutorial 7

6.042 - March 21 and March 31, 2003

These notes cover three problems that are complicated, difficult, and use most of the tools presented in class so far. There is probably much more here than you can cover, so pick and choose.

- Every man a king.
- The stop game.
- Never two kings.

(Next time, we can examples with female characters. For now, let's stick with the "king" terminology to avoid confusion.)

1 Every Man a King

Recall the definition of a *tournament*:

- There are n vertices, $p_1, \ldots, p_n$, that represent players.
- For every pair of distinct players p_i and p_j, either there is an edge from p_i to p_j (denoted $p_i \to p_j$) or there is an edge from p_j to p_i (denoted $p_j \to p_i$) but not both.
- There are no self-loops, edges $p_i \to p_i$.

We also defined the notion of a *king* in a tournament.

- We say that p_i *virtually defeats* p_j (denoted $p_i \rightsquigarrow p_j$) if player i beats player j:

  $$p_i \to p_j$$

  or if player i beats someone else who in turn beats player j:

  $$\exists p_k \; p_i \to p_k \to p_j$$

- Player p_i is a *king* if he virtually defeats every other player: $\forall j \neq i \; p_i \rightsquigarrow p_j$.

In lecture, we saw that every tournament has at least one king. Can *everyone* in an n-player tournament be a king?

1.1 Small Cases

A standard approach is to begin by considering small cases. Have students try to construct an n-player tournament where every player is a king for $n = 2$ through $n = 6$ or so. I think $n = 2$ is impossible, $n = 3$ is possible, $n = 4$ is impossible, and $n = 5$ is possible. It is tempting to conjecture that every player can be a king if and only if n is odd!

1.2 The Probabilistic Method

There is another standard approach to a question of the form:

"Does there exist a THING with a certain PROPERTY?"

In our case, the THING is an n-player tournament and the PROPERTY is that every player is a king. The basic idea of the *probabilistic method* is:

1. Construct a *random* THING.
2. Show $\Pr\{\text{THING has PROPERTY}\} > 0$.
3. Conclude that a THING with the PROPERTY must exist!

The sample space for the experiment in the first step is a set of THINGS. Therefore, if the probability that a random THING has the PROPERTY is greater than zero, then there must exist at least one THING in the sample space that has the PROPERTY.

1.3 Solving Every Man a King

The probabilistic works well for the Every Man a King problem. Define a *random tournament* to be tournament with edges oriented uniformly and independently at random. Here, "uniformly" means that each edge is as likely to point one way as the other.

An outline of the argument is provided below. You could ask students to fles out the details and provide hints as necessary.

1.3.1 Compute $\Pr\{\overline{p_i \rightsquigarrow p_j}\}$.

We want to compute the probability that player p_i does not virtually defeat player p_j. This happens only if all of the following occur:

- We do not have $p_i \to p_j$. This happens with probablity $1/2$.

- We do not have $p_i \to p_k \to p_j$, where p_k is one of the $n-2$ players other than p_i or p_j. For each player p_k, there is no such path with probability $\frac{3}{4}$.

These events are all independent, because they involve disjoint sets of edges and edges are oriented mutually independently. Therefore, the probability that all these events occur is:

$$\Pr\left\{\overline{p_i \leadsto p_j}\right\} = \frac{1}{2}\left(\frac{3}{4}\right)^{n-2}$$

1.3.2 Upper bound the probability that player p_i is not a king.

We use Boole's inequality, a.k.a. the union bound, which says that for events $E_1, \ldots, E_m$:

$$\Pr\left\{\bigcup_{k=1}^{m} E_k\right\} \leq \sum_{k=1}^{m} \Pr\{E_k\}$$

Player p_i is not a king if he does not virtually defeat some player.

$$\begin{aligned}
\Pr\{p_i \text{ is not a king}\} &= \Pr\left\{\bigcup_{j \neq i} \overline{p_i \leadsto p_j}\right\} \\
&\leq \sum_{j \neq i} \Pr\left\{\overline{p_i \leadsto p_j}\right\} \\
&= (n-1) \cdot \frac{1}{2}\left(\frac{3}{4}\right)^{n-2}
\end{aligned}$$

The first step uses the definition of a king. The second uses the union bound, and the third uses the previous result.

1.3.3 Upper bound the probability someone is not a king.

We use the union bound again.

$$\begin{aligned}
\Pr\{\text{some player not a king}\} &= \Pr\left\{\bigcup_{i=1}^{n} p_i \text{ is not a king}\right\} \\
&\leq \sum_{i=1}^{n} \Pr\{p_i \text{ is not a king}\} \\
&= n \cdot (n-1) \cdot \frac{1}{2}\left(\frac{3}{4}\right)^{n-2}
\end{aligned}$$

The first step is a rephrasing of the event. The second uses the union bound, and the third uses the previous result.

1.3.4 Show that every man is a king with positive probability.

Rewriting the previous bound, we have:

$$\Pr\{\text{every player is a king}\} \geq 1 - \frac{n(n-1)}{2\left(\frac{4}{3}\right)^{n-2}}$$

In general, *an exponential grows much faster than a polynomial.* Therefore, the denominator on the right gets much bigger than the numerator when n is large. Therefore, as n grows large, the fraction becomes very small and so the overall probability is greater than zero. Therefore, for all sufficiently large n, there exists a tournament in which every man is a king!

With a calculator, you can check that such a tournament exists for all $n \geq 21$. In fact, when n is is 50, there's a 99.9% chance that every player in a random tournament is a king!

2 The Stop Game

Here is how to play a card game called *Stop*.

- I shuffle a deck of cards.
- I deal cards off the top of the deck, face up, one at a time.
- At some time before the last card is dealt, you must say, "Stop!".
- I deal one more card. If it is black, you win; if it is red, you lose.

What is the best strategy for this game? Let the students play. Choose a player who did particularly well to explain his or her "winning strategy".

Intuitively, you should wait until there are more black cards than red remaining. If you then say stop, your odds of winning are greater than $\frac{1}{2}$. But when *exactly* should you say stop? Smart mathematicians have spent hours on this problem— no kidding! Two solutions are presented here, one using induction and one using bijection.

2.1 Solution by Induction

We can characterize a player's strategy by giving the probability that she says stop in every possible situation. Define:

$$S_{n,b} \quad = \quad \Pr\{\text{player says stop with } n \text{ cards left, } b \text{ of them black}\}$$

Thus, a player's strategy is just an assignment of values to the stop variables $S_{n,b}$. According to the rules, we must have $S_{1,0} = S_{1,1} = 1$; that is, if there is only one card left, the player has to say stop.

Now let's compute the probability that a player wins using a particular strategy. Define:

$$W_{n,b} \quad = \quad \Pr\{\text{player wins if } n \text{ cards are left, } b \text{ of them black}\}$$

We know the probability of winning for some special cases:

$$\begin{aligned} W_{n,0} &= 0 \\ W_{n,n} &= 1 \end{aligned}$$

If there are no black cards left, the player can't win. If only black cards are left, she can't lose. Ask the students to express $W_{n,b}$ in terms of other stop and win variables.

$$W_{n,b} \quad = \quad S_{n,b} \cdot \frac{b}{n} + (1 - Sn, b)\left(\frac{b}{n} \cdot W_{n-1,b-1} + \frac{n-b}{n} \cdot W_{n-1,b}\right) \quad (0 < b < n)$$

This equation says that if you stop, your winning probability is just the fraction of the remaining cards that are black. If you don't stop, then there are two cases. If the next card is black, then only $n-1$ cards remain, $b-1$ of them black, and so you win with probability $W_{n-1,b-1}$. If the next card is red, then only $n-1$ cards remain, b of them black, and so you win with probability $W_{n-1,b}$.

2.2 Small Cases

Using this recurrence, we can compute the probability that a player wins Stop with a given strategy; it is just $W_{52,26}$! But how large is $W_{52,26}$ if we use the very best strategy?

Ask students to compute $W_{2,1}$ and $W_{3,2}$ using the recurrence. Tell them to leave their answers in terms of the stop probabilities $S_{n,b}$. In fact, these cancel out.

$$
\begin{aligned}
W_{2,1} &= S_{2,1} \cdot \frac{1}{2} + (1 - S_{2,1}) \left(\frac{1}{2} \cdot W_{1,0} + \frac{1}{2} \cdot W_{1,1} \right) \\
&= S_{2,1} \cdot \frac{1}{2} + (1 - S_{2,1}) \left(\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 \right) \\
&= \frac{1}{2} \\
W_{3,2} &= S_{3,2} \cdot \frac{2}{3} + (1 - S_{3,2}) \left(\frac{2}{3} \cdot W_{2,1} + \frac{1}{3} \cdot W_{2,2} \right) \\
&= S_{3,2} \cdot \frac{2}{3} + (1 - S_{3,2}) \left(\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1 \right) \\
&= \frac{2}{3}
\end{aligned}
$$

Apparently, the player's probability of winning *does not depend on her strategy!* In fact, her probability of winning seems to be b/n, the fraction of black cards in the deck at the start. Ask students to prove this by induction. Make sure that their induction hypothesis is suitable!

Theorem 1 *If Stop is played with a deck of $n \geq 1$ cards, of which b areblack, then the player wins with probability b/n, regardless of her strategy.*

Proof. The proof is by induction on n. Let $P(n)$ be the proposition that for all strategies and all b such that $0 \leq b \leq n$, the player wins with probability b/n.

First, we show that $P(1)$ is true. In this case, only one card remains, and so the player must say stop immediately. Her probability of winning is then b/n. If $b = 0$, she always loses; if $b = 1$, she always wins.

Next, we assume $P(n-1)$ in order to prove $P(n)$. If there are no black cards left, then the player wins with probability $b/n = 0/n = 0$. If only black cards are left, then the player wins with probability $b/n = n/n = 1$. Otherwise, the player wins with probability:

$$
\begin{aligned}
W_{n,b} &= S_{n,b} \cdot \frac{b}{n} + (1 - Sn, b) \left(\frac{b}{n} \cdot W_{n-1,b-1} + \frac{n-b}{n} \cdot W_{n-1,b} \right) \\
&= S_{n,b} \cdot \frac{b}{n} + (1 - Sn, b) \left(\frac{b}{n} \cdot \frac{b-1}{n-1} + \frac{n-b}{n} \cdot \frac{b}{n-1} \right) \\
&= S_{n,b} \cdot \frac{b}{n} + (1 - Sn, b) \cdot \frac{b}{n} \\
&= \frac{b}{n}
\end{aligned}
$$

The first equation is our recurrence. The second step uses the induction hypothesis, $P(n-1)$. The remaining two steps are simplifications.

Thus, the claim holds by the principle of induction. □

2.3 A Bijection Approach

There is an another way to see that the player's strategy is irrelevant.

Suppose that we change the last rule slightly: now I deal the last card from the bottom of the deck. Say that the original rules define "game 1" and the new rules define "game 2". Are these games different? Could you, as a player, determine whether you were playing game 1 or game 2 if you didn't see the origin of the last card? How can we rigorously prove that these games are equivalent? We can use a bijection!

Lemma 2 *For every card t and sequence of cards $t_1, \ldots, t_k$, the probability that the next card dealt is t, given that cards $t_1, \ldots, t_k$ have already been dealt, is the same in games 1 and 2.*

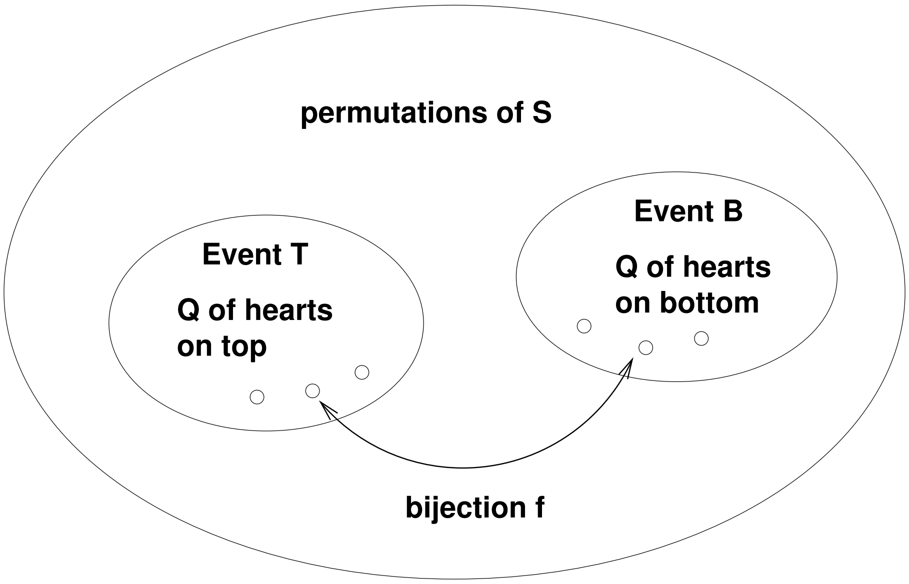
Proof. For concreteness, suppose that t is the queen of hearts. Our argument applies equally well to every other card.

Until the player says stop, the rules of the two games are identical. Therefore, the probability of dealing the queen of hearts, given that $t_1, \ldots, t_k$ have already been dealt, is the same in both games.

When the player says stop, the games diverge. In game 1, the next card comes from the top of the deck. In game 2, the next card comes from the bottom. If the queen of hearts has already been dealt, then it is the next card with probability zero in both games. Otherwise, if there is only one card left, then it is the queen of hearts with probability 1 in both games.

So suppose there is a set S of at least two cards remaining, one of which is the queen of hearts. Assuming the deck is thoroughly shuffled, so that every permutation of the remaining cards is equally likely. Then the permutations of S form a uniform sample space.

Let T be the event that the queen of hearts is on top, and let B be the event that the queen is on the bottom. These events are disjoint, because the queen can't be in both places at once:



We must show that the probability of T is equal to the probability of B. That is, the probability of dealing the queen in game 1 is equal to the probability of dealing the queen in game 2. We can do this by establishing a *bijection* between events T and B. This will prove that T and B contain an equal number of outcomes. Since all outcomes (permutations of the remaining cards) are equally likely, this proves that $\Pr\{T\} = \Pr\{B\}$. (Ask students to suggest a bijection; they might give the one below or suggest reversing the order of the remaining cards.)

Let f be a function that swaps the top and bottom cards. If $s_1, \ldots, s_n$ is the sequence of remaining cards, then f is defined as follows:

$$f(s_1 s_2 \ldots s_{n-1} s_n) \quad = \quad s_n s_2 \ldots s_{n-2} s_1$$

If the queen was on top in the permutation on the left, then the queen is on the bottom in the permutation on the right. So this is a function from T to B. Futhermore, this is a bijection, because the function is invertible; in fact, f is it's own inverse. $\square$

We've shown that games 1 and 2 are indistinguishable to the player; given the cards already dealt, each card is as likely to be dealt next in game 1 as in game 2. Therefore, in particular, every strategy the player uses in the first game works equally well in the second game.

But the second game is easy to analyze! The player wins if an only if the bottom card was red after the initial shuffle! Therefore, the player wins the second game with probability 1/2 regardless of her strategy. By our lemma, the same is true of the first game.

3 Never Two Kings

Challenge students to prove the following claim:

Claim 3 *A tournament can not contain* exactly *two kings.*

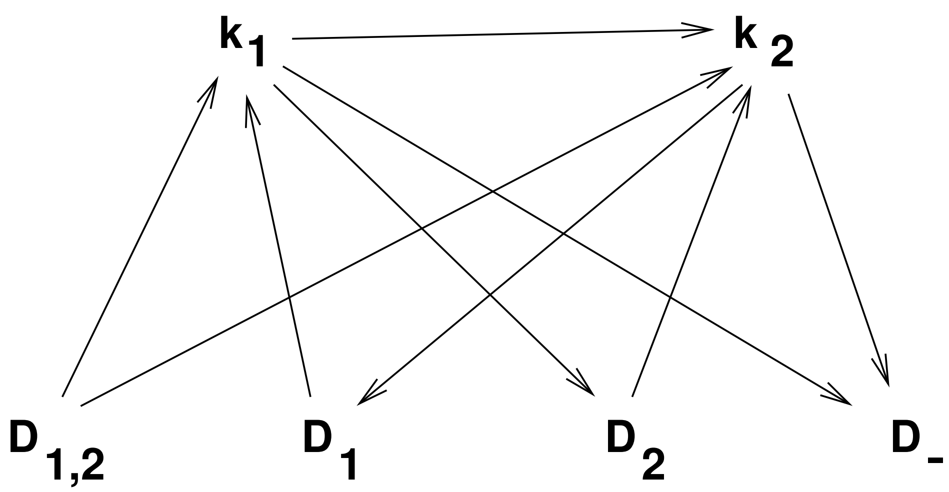
First, get them to suggest general approaches. (Here we show that if there are two kings, then there must be a third; perhaps some other approach is possible.) As a hint, suggest that they divide all other players into four groups: those that lost to both k_1 and k_2, those that beat k_1 and lost to k_2, etc. As another hint, remind them that they can use the King Chicken Theorem from lecture, which says that every tournament has a king.

Proof. Suppose that a tournament contains two kings. Let k_1 be the winner of the match between them, and let k_2 be the loser. We show that the tournament must contain a third king, k_3, as well. Therefore, no tournament contains exactly two kings.

Divide the players (excluding k_1 and k_2) into four groups:

$$
\begin{aligned}
D_{1,2} &= \text{players that defeated both } k_1 \text{ and } k_2 \\
D_1 &= \text{players that defeated only } k_1 \\
D_2 &= \text{players that defeated only } k_2 \\
D_{-} &= \text{players that defeated neither king}
\end{aligned}
$$

These definitions are illustrated below. An arrow between two groups in the diagram below indicates that *all* edges connecting those groups are oriented in the direction of the arrow.



First, suppose that $D_{1,2}$ is nonempty. Then let k_3 be the king of the tournament among the players in $D_{1,2}$. This means that k_3 virtually defeats all the players in $D_{1,2}$. Since k_3 defeats k_1, he also virtually defeats all players in D_2 and D_-. Since k_3 defeats k_2, he also virtually defeats all players in D_1. Therefore, k_3 is a king.

Otherwise, suppose that $D_{1,2}$ is empty. Since k_2 virtually defeats k_1, there must be a player x such that $k_2 \to x \to k_1$. This player x must be in D_1, and so the set D_1 is nonempty. Let k_3 be the king of the tournament among the players in D_1. Then k_3 virtually defeats all players in D_1. Player k_3 also virtually defeats both kings, since $k_3 \to k_1 \to k_2$. Since k_3 defeats k_1, he virtually defeats all players in D_2 and D_-. Since the set $D_{1,2}$ is empty, k_3 is a king.

Therefore, in either case, the tournament contains a third king. $\square$