Notes for Tutorial 5

6.042 - March 7 and March 10, 2003

There are three topics.

- Address questions and concerns about the quiz.
- Go over some examples of proofs and false proofs.
- A bit about the relevance of logic notation.

For the first item, see the quiz solution set. For the second, I've put the problems on separate pages again, so you can photocopy and distribute them if you want to. Comments on the problems come after all of them. The third section has a couple easy problems that you could put on the board.

1 Proofs and False Proofs

Challenge students to find the errors, if any, in the proofs on the following pages. Inevitably, they'll find many points of unclarity or debatable validity beyond those intended. Probably good to encourage thoughtful discussion about those objections.

1.1 The Elephant and The Mouse

Theorem 1 *The weight of a mouse is equal to the weight of an elephant.*

Proof. Let m be the weight of a mouse, and let e be the weight of an elephant. Let v be the average of their weights; that is, let $v = (m + e)/2$. Now we reason as follows:

$$\begin{aligned}
2v &= m + e && (1)\\
2v - m &= e && (2)\\
m &= 2v - e && (3)\\
(2v - m)m &= (2v - e)e && (4)\\
2vm - m^2 &= 2ve - e^2 && (5)\\
m^2 - 2vm &= e^2 - 2ve && (6)\\
m^2 - 2vm + v^2 &= e^2 - 2ve + v^2 && (7)\\
(m - v)^2 &= (e - v)^2 && (8)\\
m - v &= e - v && (9)\\
m &= e && (10)
\end{aligned}$$

In the first step, we multiply both sides of the definition of v by 2. Equations 2 and 3 follow from equation 1 by rearranging terms. Equation 4 is obtained by multiplying equations 2 and 3. In equation 5, we expand both sides. Negating both sides gives equation 6, and adding v^2 to both sides gives equation 7. The final steps are simplification; we factor both sides, take square roots, and add v to both sides. We conclude that a mouse and an elephant weigh exactly the same. □

1.2 An Inequality

Theorem 2 *For all nonnegative real numbers a and b, the following inequality holds:*

$$\sqrt{ab} \leq \frac{a+b}{2}$$

Proof. We reason as follows:

$$\begin{aligned}
\sqrt{ab} &\stackrel{?}{\leq} \frac{a+b}{2} \\
2\sqrt{ab} &\stackrel{?}{\leq} a+b \\
4ab &\stackrel{?}{\leq} a^2+2ab+b^2 \\
0 &\stackrel{?}{\leq} a^2-2ab+b^2 \\
0 &\leq (a-b)^2
\end{aligned}$$

In the first step, we multiplied both sides by 2. Then we square both sides. In the third step, we subtract $4ab$ from both sides. Then we rewrite the right side. It is easy to see that the last inequality holds, because the square of a real number can not be negative. $\square$

1.3 Roots

Theorem 3 $1 = -1$

Proof. We transform the left side into the right as follows:

$$\begin{aligned}
1 &= \sqrt{1} && (1)\\
&= \sqrt{-1 \cdot -1} && (2)\\
&= \sqrt{-1} \cdot \sqrt{-1} && (3)\\
&= \left(\sqrt{-1}\right)^2 && (4)\\
&= -1 && (5)
\end{aligned}$$

All steps are elementary. □

1.4 Too Many Primes

Theorem 4 *There are infinitely many primes.*

Proof. Suppose that the claim is false. Then there exists some finite number of primes n. Call these primes $p_1, p_2, \ldots, p_n$. Consider the quantity $m = p_1 p_2 \cdots p_n + 1$. Let q be a prime factor of m. (If m is prime, then we can choose $q = m$.) Then q can not be equal to p_i for any i, since q divides m evenly, but dividing m by p_i leaves a remainder of 1. Therefore, q is an $(n+1)$-st prime, contradicting our supposition that there are only n primes. Therefore, there are infinitely many primes. $\square$

1.5 Four Men and a Boat

Four castaways must escape from an island to the mainland on a makeshift raft. Their weights are 50, 100, 200, and 250. The raft can hold at most two people. The number of minutes required to paddle the raft between the island and the mainland is equal to the weight of the heaviest passenger aboard.

Example: If 100 and 250 cross together, 250 minutes are used. If 100 then returns alone, another 100 minutes are used.

Claim 5 *The four castaways can not all escape in less than 650 minutes.*

Proof. At least three trips and two return trips are required. Since the lightest person has weight 50, a minimum of 100 minutes are required for the two return trips. The three forward trips then require a total of 550 minutes: 100 minutes for 50 and 100, 200 minutes for 50 and 200, and 250 minutes for 50 and 250. Therefore, the total time for all four people to cross is at least $100 + 550 = 650$ minutes as claimed. □

1.6 A Bag of Numbers

Theorem 6 *Suppose that a bag contains 25 slips of paper, each marked with an integer in the range 1 to a million. Then there are two disjoint subsets of slips whose numbers have the same sum.*

Proof. There are $2^{25} - 1 \approx 32$ million different, nonempty subsets of slips. The sum of the numbers on every subset is at most 25 million. Therefore some two subsets must have the same sum. If these subsets are not disjoint, throw out the slips in common. What remains are two disjoint subsets of slips whose numbers have the same sum. $\square$

2 Comments on the Problems

2.1 The Elephant and the Mouse

The error is the step from equation 8 to equation 9. The fact that $a^2 = b^2$ does not imply $a = b$. It could be the case (and it is here) that $a = -b$.

2.2 An Inequality

This "proof" is somewhere between rotten and wrong. The problem is that the argument works from the conclusion to a true assertion, and not the other way around. The fact that the assertion implies something true does not mean that true things imply the assertion, which is what you need to show.

Somewhere, a lot of students pick up this style of argument— reasoning backward with ? over the relation until they get a true statement. If you find yourself doing this, pound your writing hand several times with a heavy textbook to make it stop. Maybe shout "WRONG! WRONG!" with each blow.

Arguably, the proof is merely rotten, because in this case, the argument can be run in reverse to give a valid proof of the inequality. (It is called the *arithmetic - geometric mean inequality*, because one side is the arithmetic mean of a and b and the other side is the geometric mean). But putting the steps in the right order is the proof-writer's job. A collection of cryptic hints as to why the conclusion should be true is *not* a proof!

2.3 Roots

The error is in the step from equation 2 to equation 3. The rule $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ does not always hold when a and b are allowed to be negative.

2.4 Too Many Primes

This is a valid proof. We showed that:

$$\text{finitely many primes} \implies \text{false conclusion}$$

In order for this statement to be true, the hypothesis must be false; that is, there can not be finitely many primes.

This is a pretty common style of proof. The general idea is that you *assume* initially assume the negation of your proposition. Then you reason from that until you reach a false conclusion. This implies that your initial assumption was wrong, and thus your proposition was true. Rosen contains some examples of this as well; look in the index under "proof, by contradiction".

2.5 Four Men and a Boat

This illustrates sloppy case analysis. By not being systematic, we miss the possibility that 10 makes a return trip. Optimal:

50, 100 cross, 100 returns	$100 + 100 = 200$ minutes
200, 250 cross, 50 returns	$250 + 50 = 300$ minutes
50, 100 cross	100 minutes
TOTAL	600 minutes

Intuitively, as long as we need 250 minutes to get the 250 guy across, we might as well get the 200 guy across at the same time.

2.6 A Bag of Numbers

This is a valid proof that foreshadows the *pigeonhole principle.* You might just perform the experiment— have each student write four or five numbers. (Warn them to make them "random".) Make the claim of the theorem. Afterward, put the slips of paper out on the table, and challenge students to find the two disjoint subsets with the same sum. While in principle they ought to be able to prove the claim for a specific set of numbers by construction, it is *very* hard. And yet we know, by this nonconstructive proof, that it is true!

3 Logic Manipulation

The symbols $\wedge$, $\vee$, $\neg$, and so on may seem very abstract. However, you'll deal with equivalent symbols all the time in programming lanuages such as C, C++, and Java. Here is a conversion table:

and	$p \wedge q$	p && q
or	$p \vee q$	p || q
not	$\neg p$	$!p$
equivalence	$p \leftrightarrow q$	$p == q$
implication	$p \rightarrow q$	$!p$ || q

These programming languages lack a symbol for the implication connective. However, it turns out that:

$$p \to q \quad = \quad \neg p \lor q$$

We can see this in the truth tables:

p	q	$p \to q$	$\neg p \lor q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

Therefore, we can write $p \to q$ in Java as $!p \mid\mid q$.

A common challenge while programming is writing complicated conditions in a simple way. This is equivalent to the problem of simplifying propositions. The same problem comes up when you're trying to simplify a digital circuit.

Can you find nice or interesting ways to rewrite the following expressions using traditional logic notation?

- $(p \land q) \lor (\neg q \land p) \lor (\neg p \land \neg q)$
- $(r \land \neg p) \lor (r \land \neg q) \lor (\neg r \land p) \lor (\neg r \land q)$

The first is just $\neg q \lor p$.

The second can be written in a million ways. For example:

$$(p \to q) \land (q \to r) \land (r \to p)$$

$$(p \land q \land r) \lor (\neg p \land \neg q \land \neg r)$$