Notes for Tutorial 2

6.042 - February 14 and 18, 2003

There are four topics.

- Proving set identities.
- Generalized Monty Hall
- The basis for tree diagrams.
- Bayes' rule

1 Proving Set Identities

In the homework, we gave you the following problem:

Let A, B, and C be sets. Assume that $A \cap B \cap C = \emptyset$. Prove that:

$$A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A)$$

We didn't tell you how to approach this problem, and we got a lot of different answers!

1.1 Venn Diagrams and the Meaning of "Proof"

One common approach was to use Venn diagrams as shown in Figure 1. You draw three overlapping circles and shade in the regions covered by the expression on the left. You draw three more circles and shade in the regions covered by the expression on the right. Then you note that these diagrams differ only in the center region, which is empty by assumption. Therefore, we have equality!

One can argue whether or not a Venn diagram argument constitutes a proof. One camp says, "A proof is a sequence of assertions, each of which follows from axioms or previous assertions via inference rules." To this camp, a picture is obviously not a proof. Where are the axioms? Where are the rules of inference? A Venn diagram is just a bunch of squiggly lines. There is good reason to distrust proofs based on pictures: they are often wrong. There is one famous argument based on pictures— Kempe's purported proof of the four-color theorem— that was accepted for 11 years before someone pointed out an error was not evident in the pictures.

Some other mathematicians say that a valid proof has to be *really* formal: reams of mathematical symbols. English is for poetry, not for rigorous mathematics. These people may

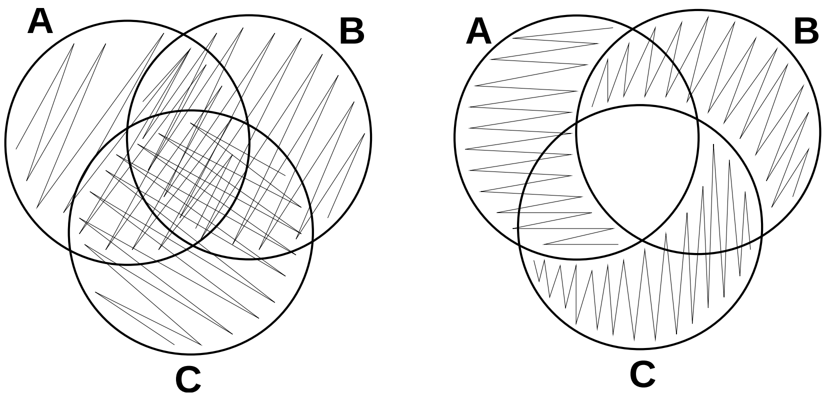

Figure 1: A Venn Diagram approach to the set identity problem.

have a point. In certain areas of research, such as the analysis distributed network protocols, such rigor seems to be essential; time and again, informal arguments have overlooked quirky special cases, resulting in buggy protocols. But in most situations, most mathematicians would say, "We could translate our proofs into ultra-formal style anytime we want to, and so our less-formal proofs are valid. But we don't want to use so much formalism, because then reading proofs would become so difficult that the field could not advance."

On the other hand, some might argue, "Yes, but we could also translate a Venn diagram into a formal proof anytime we want to, and so Venn diagram arguments are valid proofs. Furthermore, a Venn diagram is instantly convincing, but a formal proof might require ten minutes to methodically check."

You can decide where you stand on the issue of proof formality. But this is an introductory course. We do not take it as a given that you can translate a Venn diagram into a traditional proof anytime you want to. If you're asked to prove a set identity on an exam, a Venn diagram is great for clarifying your own thinking, but *is not considered a valid proof.* We want sentences and equations, not pictures.

1.2 A Traditional Proof

The proof of the set identity given below is lifted straight from the solution set. The important thing to take away is the method.

Let S and T be sets, and suppose that we want to prove that $S = T$. To do this, it is sufficient to show that *each set contains the other.* Therefore, a proof using this method contains *two parts*: first, we prove $S \subseteq T$ and then we prove $T \subseteq S$. Each of these two parts is handled in the same basic way: to prove that one set is contained in a second set, we take an arbitrary element of the first set and prove that it is also contained in the second. Often this requires quite a bit of case analysis, as in proof below.

Proof. We prove the equality by showing that the left set contains the right set and vice versa. Let S be the set on the left, $A \cup B \cup C$, and let T be the set on the right, $(A - B) \cup (B - C) \cup (C - A)$. In these terms, our objective is to prove that $S = T$.

First, we show that S is contained in T. Let x be an element of $S = A \cup B \cup C$. There are then six cases:

- $x \in A, x \notin B, x \notin C$: Since $x \in A$ and $x \notin B$, we have $x \in A - B$, and so $x \in T$.
- $x \notin A, x \in B, x \notin C$: Since $x \in B$ and $x \notin C$, we have $x \in B - C$, and so $x \in T$.
- $x \notin A, x \notin B, x \in C$: Since $x \in C$ and $x \notin A$, we have $x \in C - A$, and so $x \in T$.
- $x \in A, x \in B, x \notin C$: Since $x \in B$ and $x \notin C$, we have $x \in B - C$, and so $x \in T$.
- $x \in A, x \notin B, x \in C$: Since $x \in A$ and $x \notin B$, we have $x \in A - B$, and so $x \in T$.
- $x \notin A, x \in B, x \in C$: Since $x \in C$ and $x \notin A$, we have $x \in C - A$, and so $x \in T$.

In every case, $x \in T$ as well; therefore, every element of S is also an element of T, and so $S \subseteq T$. (Note that the case $x \in A, x \in B, x \in C$ does not exist, since $A \cap B \cap C = \emptyset$ by assumption.)

Now we show that T is contained in S. Let x be an element of $T = (A - B) \cup (B - C) \cup (C - A)$. There are then three cases:

- $x \in A - B$: This implies that $x \in A$, and so $x \in S$.
- $x \in B - C$: This implies that $x \in B$, and so $x \in S$.
- $x \in C - A$: This implies that $x \in C$, and so $x \in S$.

In every case, $x \in S$ as well; therefore, every element of T is also an element of S, and so $T \subseteq S$. Since set S contains set T and vice versa, S and T must be the same set; that is, $S = T$. $\square$

This method for proving two sets equivalent is also described in Rosen, and you can find an additional example there.

1.3 Other Methods

Set identities can be proved in other ways as well. For example, Rosen suggests transforming one side into the other using previously-established set identities. This is a completely valid type of proof that you should feel free to use.

Rosen also suggests using "membership tables". They're essentially a careful, tabular way of presenting a Venn diagram argument. They have the advantage that they work equally well with identities involving more than three sets. On the other hand, they don't demonstrate that you know how to write a traditional proof any more than a Venn diagram does. So we prefer that you not use table-based arguments on exams without a supporting explanation.

We'll talk more about what "proof" means as the term goes on. And, in any case, on quizzes we'll try to be very clear about what is acceptable as a proof and what is not. If you're not sure, ask!

2 Generalized Monty Hall

In lecture, we played the *Let's Make a Deal* game, which works as follows. You are shown three doors. You know there is a prize behind one door and there are goats behind the other two. You pick a door. To build suspense, Carol always opens a *different* door, revealing a goat. You can then stick with your original door or switch to the other unopened door. You win whatever is behind the door you now select. We showed that you should always switch to the other door; by doing so, you win the prize with probability $\frac{2}{3}$.

The game we played in lecture differed from the television game show in several respects. On the game show, Monty did not always reveal a goat and give the contestant the option to switch doors. Sometimes the contestant was immediately awarded whatever lay behind the door he picked initially.

This suggests a new game, which we might call *generalized Monty Hall*. As before, you are shown three doors. You know there is a prize behind one door and there are goats behind the other two. You pick a door. Now, at Monty's discretion, one of two things happens:

1. Carol opens the door you picked and you win whatever lies behind it.
2. Carol opens a different door to reveal a goat. You can then stick with your original door or switch to the other unopened door. You win whatever is behind the door you now select.

Suppose that you are playing this game. What should you do in the latter case, where you're given the option to stick with your current door or to switch to the other?

In order to analyze this problem, let's introduce some variables that capture the range of possible strategies for Monty and yourself:

- Let r be the probability that Monty reveals a goat and lets you switch when your initial guess is *right*.
- Let w be the probability that Monty reveals a goat and lets you switch when your initial guess is *wrong*.

- Let p be the probability that you switch when given the option to do so.

Now we can compute the probability of the event that you win the game using a tree diagram. Since the tree is rather large, we've drawn only one of the three main subtrees in Figure 2. Fortunately, the other two subtrees are symmetric to the first; only the door numbers are permuted. Therefore, we can compute the probability of the event that you win by adding the probabilities of winning outcomes in the first subtree and multiplying the answer by 3.

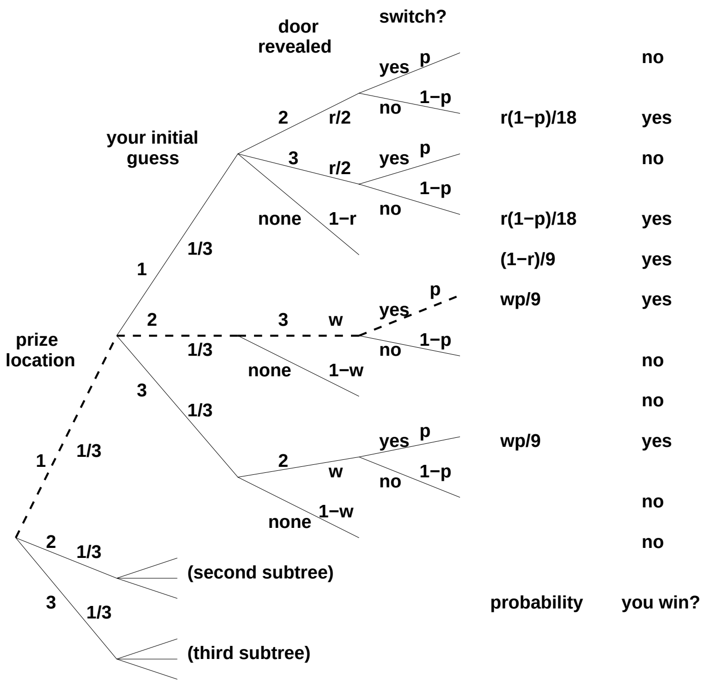

Figure 2: A tree diagram for the generalized Monty Hall game.

From the tree diagram, we have:

$$\begin{aligned} \Pr\{\text{you win}\} &= 3 \cdot \left(\frac{r(1-p)}{18} + \frac{r(1-p)}{18} + \frac{1-r}{9} + \frac{wp}{9} + \frac{wp}{9} \right) \\ &= \frac{1 + p(2w - r)}{3} \end{aligned}$$

Let's interpret this solution by considering some special cases.

- If you never switch ($p = 0$), then your probability of winning is $\frac{1}{3}$, regardless of Monty's strategy.

- If Monty always gives you the option to switch ($r = w = 1$) and you always do ($p = 1$), then your probability of winning is $\frac{2}{3}$. This is precisely what our analysis in lecture showed.
- Suppose that you employ the "switch" strategy ($p = 1$), but Monty is trying to make you lose. Then he can give you the option to switch only when your initial guess is right ($r = 1, w = 0$). In this case, your probability of winning is zero!
- On the other hand, suppose that you employ the "switch" strategy ($p = 1$), but Monty is trying to make you win. Then he can give you the option to switch only when your initial guess is wrong ($r = 0, w = 1$). Then you always win!

So what is the solution? If you're on the game show, what do you do? By always sticking, you can assure yourself a 1/3 chance of winning, regardless of what Monty does. The rest is psychology; probability does not give us the answer!

- If you think Monty is trying to make you lose, never switch.
- If you think Money trying to help you win, always switch.

There are simpler games that come down to psychology in this way. Suppose I borrow your wallet and count how much money is inside. Then I say, "Want to switch the money in my wallet for the money in your wallet?!" If you think I'm out to cheat you, then you should say "no". After all, if I had more money in my wallet, I wouldn't have offered to switch! On the other hand, if you think I'm a nice guy trying to give you a leg up in this harsh world, then you should say "yes". If you want to be conservative and avoid mind games, stick with your own wallet. Probability doesn't give us a full answer in this simple game in the same sense that it doesn't tell us the whole story in Monty Hall.

3 Tree Diagrams and Conditional Probability

We talked some big talk about how Venn diagrams are just a bunch of squiggles and how we want mathematics not pictures. This is somewhat awkward, because we've been using tree diagrams— mere pictures!— to work out probability problems all along. In particular, we've asserted that the probability of an outcome is the product of the edge probabilities on the path from the root to that outcome. But why should this be true? For example, in Figure 2, we concluded that the probability of the outcome that the prize is behind door 1, your initial guess is door 2, door 3 is revealed, and then you switch is equal to the product of edge probabilities on the highlighted path:

$$\frac{1}{3} \cdot \frac{1}{3} \cdot w \cdot p \quad = \quad \frac{wp}{9}$$

Let's at least understand the mathematical justification for this conclusion! Define the following events:

- Let A be the event that the prize is behind door 1.
- Let B be the event that your initial guess is door 2.
- Let C be the event that Carol reveals door 3.
- Let D be the event that you switch.

Our goal is to find the probability of the outcome in which all these events occur, namely $\Pr\{A \cap B \cap C \cap D\}$.

We can interpret the edge probabilities along the highlighted path in Figure 2 in terms of these variables.

- The first edge probability on the highlighted path, $\frac{1}{3}$, is simply $\Pr\{A\}$, the probability that the prize is behind door 1.
- The second edge probability, $\frac{1}{3}$, is $\Pr\{B \mid A\}$, the probability that your initial guess is door 2, given that the prize is behind door 1.
- The third probability, w, is $\Pr\{C \mid A \cap B\}$, the probability that Carol reveals door 3, given that the prize is behind door 1 and your initial guess is door 2.
- The final edge probability, p, is $\Pr\{D \mid A \cap B \cap C\}$, the probability that you switch, given that the prize is behind door 1, your initial guess is door 2, and Carol reveals door 3.

In general, we can interpret an experiment as a random walk from the root of a tree diagram to one of the leaves. Under this interpretation, the probability on an edge in a tree diagram is the conditional probability that you traverse that edge, given that you reach its parent node in the tree.

Now let's dinf out what the product of the edge probabilities really is:

$$\begin{aligned}
&\Pr\{A\} \cdot \Pr\{B \mid A\} \cdot \Pr\{C \mid A \cap B\} \cdot \Pr\{D \mid A \cap B \cap C\} \\
&\qquad = \Pr\{A\} \cdot \frac{\Pr\{B \cap A\}}{\Pr\{A\}} \cdot \frac{\Pr\{C \cap A \cap B\}}{\Pr\{A \cap B\}} \cdot \frac{\Pr\{D \cap A \cap B \cap C\}}{\Pr\{A \cap B \cap C\}} \\
&\qquad = \Pr\{A \cap B \cap C \cap D\}
\end{aligned}$$

The first step uses the definition of conditional probability, and the second uses only cancellation of terms.

Amazingly enough, the product of the conditional probabilities on the edges of the tree diagram is exactly the outcome probability that we were looking for! The method of multiplying edge probabilities in a tree diagram to compute outcome probabilities can be justified in general by extending the argument above.

With this formal justification in hand, you are free to continue using tree diagrams in your solutions and proofs in this class.

4 Bayes' rule

What is the relationship between $\Pr\{A \mid B\}$ and $\Pr\{B \mid A\}$? Are they the same thing? Can you give an example where it is clear that these probabilities are different? Some simple examples suggest that the answer is "no". For example, the probability that you are a female given that you are an MIT student is now about $\frac{1}{2}$. On the other hand, the probability that you are an MIT student given that you are a female is very small. But we can write one conditional probability in terms of the other pretty easily:

$$\Pr\{B \mid A\} = \Pr\{A \mid B\} \cdot \frac{\Pr\{B\}}{\Pr\{A\}}$$

This is called *Bayes' rule* and is a neat way of turning around cause and effect. Here A and B are events with nonzero probability. You can verify the equation above for yourself using the definition of conditional probability.

Let's work an example that uses Bayes' rule. A weatherman walks to work each day. Some days it rains:

$$\Pr\{\text{rains}\} = 0.30$$

Sometimes the weatherman brings his umbrella. Usually this is because he predicts rain, but he also sometimes carries it to ward off bright sunshine.

$$\Pr\{\text{carries umbrella}\} = 0.40$$

As a weatherman, he usually doesn't get caught out in a storm without protection:

$$\Pr\{\text{carries umbrella} \mid \text{rains}\} = 0.80$$

Suppose you see the weatherman walking to work, carrying an umbrella. What is the probability of rain?

$$\begin{aligned}
\Pr\{\text{rains} \mid \text{carries umbrella}\} &= \Pr\{\text{carries umbrella} \mid \text{rains}\} \cdot \frac{\Pr\{\text{rains}\}}{\Pr\{\text{carries umbrella}\}} \\
&= 0.80 \cdot \frac{0.30}{0.40} \\
&= 0.60
\end{aligned}$$

We're turned around cause and effect. Risk of rain has the effect of making the weatherman carry his umbrella. Yet we've shown that if he carries his umbrella, it is pretty likely to rain!