Notes for Tutorial 12

6.042 - May 2 and May 5

There are three main topics. An understanding of this stuff should translate pretty directly to a higher final exam score.

- More counting practice.
- Inclusion-exclusion
- Combinatorial proof

In addition, there is a brief brush with probability distributions, a topic that we won't cover in detail. (So far as I know...)

1 More Counting Practice

As before, I suggest you pose these questions to the group. Have people explain their answers so that everyone understands and can ask questions.

1. How many ways are there to arrange the letters in the word "bumblebee"?

$$\binom{9}{3,3,1,1,1} = \frac{9!}{3!\ 3!\ 1!\ 1!\ 1!}$$

   First, suppose that the letters were all distinct, as in $b_1umb_2le_1b_3e_2e_3$. Then there would be 9! arrangements. However, "b" is repeated three times and "e" is repeated three times. Therefore, there is a $(3! \cdot 3!)$-to-1 mapping from arrangements of letters in the subscripted word to arrangements of letters in the ordinary word. (The mapping is performed by erasing the subscripts.) Thus, the division rule gives the answer above.

2. How many ways are there to arrange the letters in the word "booboo"?

   (Call on a student who was confused on the previous question, if there was one.) By the same reasoning as before:

$$\binom{6}{4,2} = \frac{6!}{4!\ 2!}$$

3. In how many ways can 20 identical balls be arranged in 5 distinguishable bins?

   There is a bijection with sequences containing 20 stars and 4 bars, so we get:

$$\binom{(20+5-1)!}{20!\ (5-1)!}$$

4. How many sequences are there containing 12 stars and 4 bars?

$$\frac{(12+4)!}{12!\ 4!}$$

   Some students reflexively subtract 1 from the nearest expression whenever they see the words "stars" and "bars". Make sure understand why this was necessary in the preceeding problem, but not in this one.

5. I want to string together $k > 3$ unique beads to form a bracelet. In how many ways can this be done?

   There are $k!$ ways to line up the beads. There is a k-to-1 mapping from lineups of beads to circles of beads, as we saw in the Knights of the Round Table example in lecture. Futhermore, there is a 2-to-1 mapping from circles of beads to bracelets, because we can turn the bracelet over. Therefore, two applications of the division rule give:

$$\frac{n!}{2n} = \frac{(n-1)!}{2}$$

6. How many distinct positive factors does the number $3^4 \cdot 5^2 \cdot 7^3 \cdot 11$ have?

   A factor is a product of between zero and four 3's, between zero and two 5's, between zero and three 7's and either zero or one 11. Therefore, by the product rule, the answer is:

$$(4+1) \cdot (2+1) \cdot (3+1) \cdot 2 = 120$$

7. In how many ways can 20 identical books be placed on 5 different shelves?

   This is equivalent to asking how many ways there are to arrange 20 stars and 4 bars in a string. The answer is:

$$\frac{24!}{20!\ 4!}$$

8. In how many ways can 20 identical books be placed on 5 different shelves, if every shelf must get at least 1 book?

   First, we place one book on every shelf. Then we may place the remaining 15 books on the 5 shelves without restrictions. By a stars-and-bars argument, the number of ways to do this is:

$$\frac{19!}{15!\ 4!}$$

9. In how many ways can 20 distinguishable books be placed on 5 different shelves?

   This is equivalent to asking how many ways there are to arrange the symbols $b_1, b_2, \ldots, b_{20}$ together with four bars in a string. The answer is:

$$\frac{24!}{4!}$$

10. What is the probability of flipping 80 heads and 20 tails on a coin that comes up heads with probability 0.6?

    There are $\binom{80+20}{20}$ such sequences, and each sequence has probability $0.6^{80} \cdot 0.4^{20}$. Therefore, the probability is:

$$\binom{80+20}{20} \cdot 0.6^{80} \cdot 0.4^{20}$$

11. How many injective functions are there from an n-element domain to an m-element range?

    If $m < n$, then there are none. Otherwise, we can map the first domain element to m points in the range, the second domain element to $m-1$ points in the range, etc., until we run out of domain elements. Overall, this gives:

$$m \cdot (m-1) \cdots (m-n+1) \quad = \quad \frac{m!}{n!}$$

2 Inclusion-Exclusion

The inclusion-exclusion formula is easier to describe than to write down. It says that the size of a union of sets is equal to the sum of the set sizes *minus* the sizes of the pairwise intersections *plus* the sizes of the three-way intersections *minus* the size of the four-way intersections, etc. (You could ask three students two write up the inclusion-exclusion formulas for 2, 3, and 4 sets in parallel for reference.)

The following problems are suitable for small teams.

2.1 An easy problem

On an island, everyone is healthy, wealthy, or wise. There are exactly 1000 people with each property. There are exactly 200 people with each pair of properties. And there are exactly 50 people with all three properties. How many people are on the island?

Let H be the set of healthy people, \$ be the set of wealthy people, and W be the set of wise people. By the inclusion-exclusion formula, we have:

$$\begin{aligned}
|H \cup \$ \cup W| &= |H| + |\$| + |W| - |H \cap \$| - |H \cap W| - |\$ \cap W| + |H \cap \$ \cap W| \\
&= 3 \cdot 1000 - 3 \cdot 200 + 50 \\
&= 2450
\end{aligned}$$

2.2 A harder problem

A number is *cube-free* if it is not divisible by a perfect cube greater than 1. For example, 36 is cube-free, but $54 = 3^3 \cdot 2$ is not. Use inclusion-exclusion to count the cube-free integers in the range from 1 to 999.

Initially, we'll compute the size of the complementary set; that is, we'll count the numbers from 1 to 999 that *are* divisible by a perfect cube.

Let S_q be the set of natural numbers from 1 to 999 that are divisible by q^3. Our goal is to compute:

$$|S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6 \cup S_7 \cup S_8 \cup S_9|$$

This would be a lot of work! But notice that $S_4 \subseteq S_2$; that is, every number divisible by 4^3 is also divisible by 2^3. Since S_4 adds nothing to the union above, we can forget about it. By similar reasoning, we can reduce the problem to computing:

$$|S_2 \cup S_3 \cup S_5 \cup S_7|$$

This is more tractable. We now compute the terms of the inclusion-exclusion formula:

$$\begin{aligned}
|S_2| &= \lfloor \tfrac{999}{2^3} \rfloor &= 124 \\
|S_3| &= \lfloor \tfrac{999}{3^3} \rfloor &= 37 \\
|S_5| &= \lfloor \tfrac{999}{5^3} \rfloor &= 7 \\
|S_7| &= \lfloor \tfrac{999}{7^3} \rfloor &= 2 \\
|S_2 \cap S_3| &= \lfloor \tfrac{999}{2^3\ 3^3} \rfloor &= 4
\end{aligned}$$

All other terms are ZERO! So we have:

$$\begin{aligned} |S_2 \cup S_3 \cup S_5 \cup S_7| &= 124 + 37 + 7 + 2 - 4 \\ &= 166 \end{aligned}$$

This means that 166 numbers in the range 1 to 999 are multiples of perfect cubes. Therefore, the number of cube-free numbers in this range is:

$$999 - 166 \;=\; 833$$

3 Combinatorial Proofs

Give a combinatorial proof of the following identity:

$$(r-k) \cdot \binom{r}{k} \;=\; r \cdot \binom{r-1}{k}$$

Remember the general structure of a combinatorial proof:

1. Define an appropriate set S.
2. Show that the left side counts the number of elements in S.
3. Show that, from another perspective, the right side also counts the number of elements in S.
4. Conclude that the two sides must be equal, because they are both equal to $|S|$.

Let set S consists of all $(k+1)$-element subsets of $\{1, \ldots, n\}$ where one element of the subset is specially marked. Formally:

$$S \;=\; \{(T, x) \mid T \subseteq \{1, \ldots, n\}, |T| = k, \text{ and } x \in T\}$$

(More intuitively, S consists of all $k+1$ person committees with a designated chairman.)

On one hand, we could pick k elements in $\binom{r}{k}$ ways and then choose the special element from among the remaining elements in $(r-k)$ ways. Thus, the left side of the equation is equal to $|S|$. On the other hand, we could choose the special element first in r ways and then choose the remaining k elements in $\binom{r-1}{k}$ ways. Thus, the right side is equal to $|S|$. Since both sides are equal to $|S|$, they must be equal to each other.

3.1 A harder identity

$$n2^{n-1} \quad = \quad \sum_{k=1}^{n} k \cdot \binom{n}{k}$$

Set S consists of *all* subsets of $\{1, \ldots, n\}$ where one element is special. Formally:

$$S \quad = \quad \{(T, x) \mid T \subseteq \{1, \ldots, n\} \text{ and } x \in T\}$$

(Intuitively, S consists of all committees with one or more people and a designated chairman.)

On one hand, we could first pick one of the n elements to be special and then either include or exclude every other element. (That is, we first pick x and stick it in T. Then we decide whether or not to include every other element in T.) This can be done in $n \cdot 2^{n-1}$ ways. Therefore, the left side is equal to $|S|$.

On the other hand, our subset must contain k elements for some k between 1 and n. For each k, we can choose the subset is $\binom{n}{k}$ ways and then choose the special element from that subset in k ways. Therefore, the right side is also equal to $|S|$. Since the left and right sides of the equation are both equal to $|S|$, they must be equal to each other.

4 Distributions

Let the random variable Q be the number of 1's in an n-bit string selected uniformly at random. Let the random variable R be the number of times heads comes up on n tosses of a fair coin. From one perspective, these random variables have nothing to do with one another; after all, they are defined on different sample spaces. On the other hand, they clearly have a lot in common. In particular, these two random variables have the same *probability density function*:

The *probability density function (pdf)* of a random variable R is denoted PDF_R and defined by:

$$\mathrm{PDF}_R(x) \quad = \quad \Pr\{R = x\}$$

If we graph the probabiilty density functions for the random variables Q and R, we get something like this. [Draw a binomial distribution. It is defined only on integral points, but put a curve through those points to illustrate the shape.] This is called the *binomial distribution*.

By studying properties of the binomial distribution, we can reach conclusions about all random variables with this probability density function– and such random variables come up all the time! For example, what is the height of the graph at x?

$$
\begin{aligned}
\mathrm{P}DF_R(x) &= \Pr\{R = x\} \\
&= \frac{1}{2^n} \cdot \binom{n}{x}
\end{aligned}
$$

This is the probability that an n-bit string contains x 1's and also the probability that we flip x heads in n tosses of a fair coin. One fact about the probability density function gives an answer to multiple problems.

There's one other probability density function that we've already encountered many times. What does the probability density function for a *uniform* random variable look like? (horizontal line)