# Notes for Tutorial 10

6.042 - April 25 and April 29, 2003

There are three topics.

- Counting practice.
- Bounding a sum using integration.
- Binomial theorem and consequences.

In addition, try to weasel in the idea of combinatorial proof somewhere. There are no problems on this in the current homework, but there probably will be on the next. There are opportunities in both the counting practice and just before the proof of the binomial theorem.

## 1 Counting Practice

I have $f$ fiction books, $n$ nonfiction books. All are different. Assume $f, n \geq 5$. Ask these questions verbally or write them on the board. Have students answer aloud and explain their reasoning.

- In how many ways can I arrange these books on a shelf?

$$(f+n)!$$

- In how many ways can I arrange these books on a shelf if I keep the fiction books together and the nonfiction books together?

  Either fiction comes first or nonfiction does, so we have:

$$2 \cdot f! \cdot n!$$

- I want to bring 3 fiction books and 2 nonfiction books on a trip. In how many different ways can I choose which books to bring?

$$\binom{f}{3}\binom{n}{2}$$

- In May, I plan to read 3 different fiction books followed by 2 different nonfiction books. In how many different ways can I do this? (Here, assume that order matters; that is, reading book $A$ and then book $B$ is one possibility, but reading $B$ and then $A$ is considered a different possibility.)

$$f(f-1)(f-2)n(n-1)$$

- My friend plans to read 5 of my books, 3 of them fiction and 2 of them nonfiction. In how many different ways can she do this? (Again, assume that order matters.)

  We could select the books and then order them:

$$\binom{f}{3}\binom{n}{2} \cdot 5!$$

  Or we could order the fiction books and order the nonfiction books, and then pick 3 of the 5 "reading slots" to fill with fiction books:

$$f(f-1)(f-2)n(n-1) \cdot \binom{5}{3}$$

  These expressions must be equivalent! It is easy to check that they are.

- On another trip, I plan to bring a total of $k$ books, of which $i$ are fiction and $k-i$ are nonfiction. In how many different ways can I do this? (Order does not matter; assume $k < f$ and $k < n$.)

$$\binom{f}{i}\binom{n}{k-i}$$

- On my trip, I plan to bring a total of $k$ books. In how many different ways can I do this? (Again, order does not matter.)

  The obvious answer is:

$$\binom{f+n}{k}$$

  An alternative is to note that we must read $i$ fiction books for some $i$ in the range $0 \leq i \leq k$ and use the preceding answer:

$$\sum_{i=0}^{k}\binom{f}{i}\binom{n}{k-i}$$

Since we've counted the same thing in two different ways, it must be that:

$$\binom{f+n}{k} = \sum_{i=0}^{k} \binom{f}{i}\binom{n}{k-i}$$

We've proved that this equation holds by *counting the same thing in two different ways.* This is called a *combinatorial proof.*

## 2 Approximating a Sum via Integration

One of the functions that we'll use the most for the remainder of the term is factorial. In lecture, we saw how to upper and lower bound a sum using integration. It would be nice to get similar bounds on $n!$. Unfortunately, this is a product:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

If there were a way to make this product into a sum, we could get upper and lower bounds by integration. How could we could this? Take the logaritm of both sides:

$$\ln n! = \ln(n) + \ln(n-1) + \ln(n-2) + \cdots + \ln(3) + \ln(2) + \ln(1)$$

Ask students to compute upper and lower bounds on $n!$ using the integral method. Encourage them to draw a picture, since this really helps get the bounding functions and integration ranges right. You might provide them with the following integration formula:

$$\int \ln x \; dx = x \ln x - x$$

There are many ways to set up the integrals and handle the boundaries. Here is one way.

$$\begin{aligned} \sum_{k=1}^{n} \ln k &< \int_0^n \ln(x+1) \; dx \\ &= (x+1)\ln(x+1) - (x+1)\big|_0^n \\ &= (n+1)\ln(n+1) - n \end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{n} \ln k &> \int_1^n \ln x \; dx \\ &= x \ln x - x|_1^n \\ &= n \ln n - n + 1\end{aligned}$$

Now exponentiating gives the bounds:

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}$$

This shows that $n!$ is right around $(n/e)^n$. With a bit more work, one can obtain the following approximation, which is worth remembering:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

This looks a lot worse than $n!$, but is actually much easier to work with in practice.

For example, what is the probability that we flip *exactly* a thousand heads when we flip a fair coin two thousand times? The exact answer is:

$$\frac{1}{2^{2000}} \cdot \binom{2000}{1000} = \frac{1}{2^{2000}} \cdot \frac{2000!}{(1000!)^2}$$

Good luck computing two thousand factorial. Letting $n = 1000$ and using Stirling's formula gives:

$$\begin{aligned}\frac{1}{2^{2n}} \cdot \binom{2n}{n} &= \frac{1}{2^{2n}} \cdot \frac{(2n)!}{(n!)^2} \\ &= \frac{1}{2^{2n}} \cdot \frac{\sqrt{2\pi(2n)}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2} \\ &= \frac{1}{\sqrt{\pi n}}\end{aligned}$$

Everything cancels! We can evaluate this last expression with a calculator. The probability is $0.0178\ldots$, or about 1.8%. Thus, even though the coin is fair, it is extremely *unlikely* that you get half heads and half tails!

# 3 The Binomial Theorem

Here's a neat combinatorial fact:

**Fact 1**

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

We *can* prove this algebraically by subtituting in the definitions of the binomial coefficients and working out all the factorials. (You should probably not work through this, but here's the argument if you must.)

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!\,(n-k-1)!} + \frac{(n-1)!}{(k-1)!\,(n-k)!} \\
&= \left(\frac{1}{k} + \frac{1}{n-k}\right) \cdot \frac{(n-1)!}{(k-1)!\,(n-k-1)!} \\
&= \frac{n}{k(n-k)} \cdot \frac{(n-1)!}{(k-1)!\,(n-k-1)!} \\
&= \frac{n!}{k!\,(n-k)!} \\
&= \binom{n}{k}
\end{aligned}$$

There is an easy way to see that this equation holds. How many ways are there to choose $k$ objects from a set of $n$ objects?

$$\binom{n}{k}$$

But there's another way to answer that question. Pick one of the $n$ objects and call it $x$. Now when we choose $k$ objects from the set of $n$, we either pick $x$ or we don't. If we do pick $x$, how many ways are there to choose the remaining $k-1$ objects?

$$\binom{n-1}{k-1}$$

If we don't pick $x$, then how many ways are there to choose the $k$ objects?

$$\binom{n-1}{k}$$

Therefore, we have:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

This is called a *combinatorial proof.* The basic idea is to *count the same thing in two different ways.* This gives two expressions that must be equal. Here we've counted the number of ways to pick $k$ objects from a set of $n$ in two different ways and gotten a neat equation—without using any algebra! We'll see more of this later on.

We can use our equation and induction to prove one of the most important theorems of all time! In fact, this is where binomial coefficients get their name. (Make this an exercise.)

**Theorem 1 (Binomial Theorem)** *For all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$:*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

*Proof.* The proof is by induction on $n$. Let $P(n)$ be the proposition that:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

for all $x, y \in \mathbb{R}$. Note that $P(0)$ is true, because $(x+y)^0 = 1$ and:

$$\begin{aligned} \sum_{k=0}^{0} \binom{n}{k} x^k y^{n-k} &= \binom{0}{0} x^0 y^0 \\ &= 1 \end{aligned}$$

Now we must show that for $n \geq 1$, $P(n-1)$ implies $P(n)$. To that end, we assume $P(n-1)$ and reason as follows:

$$
\begin{aligned}
(x+y)^n &= (x+y)\cdot(x+y)^{n-1} \\
&= (x+y)\cdot\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k} \\
&= x\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k} + y\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k} \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-k} \\
&= \sum_{k=1}^{n}\binom{n-1}{k-1}x^k y^{n-k} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}
\end{aligned}
$$

The first step uses algebra. The second uses the induction hypothesis. In the third step, we use the distributive law. Next, we move $x$ and $y$ into the summations. In the fifth step, we reindex the first summation. The final step uses the identity derived above and the relations:

$$
\begin{aligned}
\binom{n-1}{n-1} &= \binom{n}{n} \\
\binom{n-1}{0} &= \binom{n}{0}
\end{aligned}
$$

Thus, the claim is proved by induction. □