Problem Set 9

Due: Start of class on Tuesday, May 6.

Problem 1. Solve the following counting problems, and briefly explain your reasoning. You may express your answers in terms of factorials and binomial coefficients.

(a) How many ways are there to arrange the letters in the word "bookkeeper"?

Solution. If all 10 letters in the word bookkeeper were distinct, then there would be 10! arrangements. However, this overcounts by a factor of 2! 2! 3!, because the two o's are indistinguishable, the two k's are indistinguishable, and the three e's are indistinguishable. Therefore, by the division rule, the number of arrangements is:

$$\frac{10!}{2!\,2!\,3!}$$

Alternatively, this follows from the formula for permutations with indistinguishable objects.

(b) How many different sequences are there that consist of 99 stars and 7 bars?

Solution. We must choose 7 of the $99 + 7 = 106$ positions in the sequence to contain bars. Therefore, the number of such sequences is:

$$\binom{99+7}{7}$$

(c) In how many different ways can $2n$ students be paired up?

Solution. Pair up students by the following procedure. Line up the students and pair the first and second, the third and fourth, the fifth and sixth, etc. The students can be lined up in $(2n)!$ ways. However, this overcounts by a factor of 2^n, because we would get the same pairing if the first and second students were swapped, the third and fourth were swapped, etc. Furthermore, we are still overcounting by a factor of $n!$, because we would get the same pairing even if pairs of students were permuted, e.g. the first and second were swapped with the ninth and tenth. Therefore, the number of pairings is:

$$\frac{(2n)!}{2^n \cdot n!}$$

(d) How many nondecreasing sequences of 7 integers in the range from 0 to 99 are there?

Solution. There is a bijection between these nondecreasing sequences and strings containing 7 bars and 99 stars. In particular, the k-th number in the nondecreasing sequence is the number of stars preceding the k-th bar. For example:

$$\star \mid \star\star\star \mid\mid \star\star \mid \star \ldots \quad \leftrightarrow \quad (1, 4, 4, 6, \ldots)$$

Therefore, the number of such nondecreasing sequences is:

$$\binom{99+7}{7}$$

(e) How many different solutions over the natural numbers are there to the following equation?

$$x_1 + x_2 + x_3 + \ldots + x_{10} \quad = \quad 1000$$

A solution is a specification of the value of each variable x_i. Two solutions are different if different values are specified for some variable x_i.

Solution. There is a bijection between strings containing 1000 stars and 9 bars and solutions to this equation. The 9 bars divide the stars into 10 segments. Let x_i be the number of stars in the i-th segment. Therefore, the number of solutions is:

$$\binom{1000+9}{9}$$

(f) In how many different ways can one choose n out of $2n$ objects, given that n of the $2n$ objects are identical and the other n are all unique?

Solution. We can select n objects as follows. First, take a subset of the unique objects. Then take however many identical elements are needed to bring the total to n. The first step can be done in 2^n ways, and the second can be done in only 1 way. Therefore, there are 2^n ways to choose n objects.

(g) How many different paths are there from point $(0, 0, 0)$ to point $(10, 20, 30)$ if every step increments one coordinate and leaves the other two unchanged?

Solution. There is a bijection between the set of all such paths and the set of strings containing 10 X's, 20 Y's, and 30 Z's. In particular, we obtain a path by working through a string from left to right. An X corresponds to a step that increments the first coordinate, a Y increments the second coordinate, and a Z increments the third. The number of such strings is:

$$\frac{60!}{10!\ 20!\ 30!}$$

Therefore, this is also the number of paths.

(h) In how many different ways can 100 identical balls be arranged in 10 different bins, if every bin must receive at least 2 balls?

Solution. There is a bijection between arrangements of balls and bins and strings containing 9 bars and 80 stars. The 9 bars partition the sequence of stars into 10 segments. If there are x_i stars in the i-th segment of a string, then there are $x_i + 2$ balls in the i-th bin of the corresponding arrangment. Therefore, the number of arrangements is equal to the number of strings, which is:

$$\binom{80+9}{9}$$

(i) How many undirected graphs without self-loops are there with vertices $v_1, v_2, \ldots, v_n$?

Solution. There are $\binom{n}{2}$ potential edges, each of which may or may not appear in a given graph. Therefore, the number of graphs is:

$$2^{\binom{n}{2}}$$

Problem 2. A pizza shop is having a special promotion. Their ad reads:

> Buy 2 large pizzas at the regular price, and add any subset of our 11 delicious toppings to each pizza at no extra cost. That's 4,194,304 different ways to design your order!

The ad writer is a former MIT student who figures that the number of possible orders is $(2^{11})^2 = 4,194,304$. Unfortunately, this student got a B- in 6.042, and his solution is wrong. What is his mistake, and how many different orders are really possible?

Solution. The number of ways to choose different toppings for one pizza is the number of subsets of the set of 11 toppings, which is 2^{11}. However, when the student says that there are $(2^{11})^2$ different ways to choose toppings for two pizzas, he is double-counting orders in which the two pizzas receive the same toppings. The right answer is:

$$2^{11} + \binom{2^{11}}{2} \quad = \quad 2,098,176$$

This is the number of ways to choose the same toppings for both pizzas plus the number of ways to choose two pizzas with different sets of toppings.

Problem 3. Let S be a set containing n elements.

(a) How many binary relations on the set S are there?

Solution. For every ordered pair of elements of S, the relation either holds or it does not. Therefore, there are 2^{n^2} binary relations.

(b) How many binary relations on S are symmetric?

Solution. We must decide whether the relation holds for every unordered pair of elements, of which there are $\binom{n}{2}$. We must also decide whether each of the n elements is related to itself. The total number of ways to take these decisions is:

$$2^{\binom{n}{2}+n}$$

(c) How many binary relations on S are reflexive?

Solution. If the relation is reflexive, then the relation may or may not hold for every ordered pair of distinct elements, of which there are $n^2 - n$. Thus, the number of reflexive relations is:

$$2^{n^2-n}$$

(d) How many binary relations on S are both symmetric and reflexive?

Solution. Now the relation may or may not hold for every unordered pair of distinct elements, of which there are $\binom{n}{2}$. Thus, the number of symmetric, reflexive relations is:

$$2^{\binom{n}{2}}$$

(e) How many binary relations on S are symmetric or reflexive?

Solution. By inclusion exclusion, we have:

$$\begin{aligned}
&\#\text{ symmetric or reflexive} \\
&\qquad = \#\text{ symmetric} + \#\text{ reflexive} - \#\text{ symmetric and reflexive} \\
&\qquad = 2^{\binom{n}{2}+n} + 2^{n^2-n} - 2^{\binom{n}{2}}
\end{aligned}$$

Problem 4. Give combinatorial proofs of the identities below. Use the following structure for each proof. First, define an appropriate set S. Next, show that the left side of the equation counts the number of elements in S. Then show that, from another perspective, the right side of the equation also counts the number of elements in set S. Conclude that the left side must be equal to the right, since both are equal to $|S|$.

(a)

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{n-k}$$

Solution. We are counting the number of $2n$-bit strings with an equal number of zeros and ones. One one hand, the number of such strings is equal to $\binom{2n}{n}$, since we must choose n of the $2n$ bits to be 1's. This gives the expression on the left side of the equation. On the other hand, the number of 1's in the first half of the string must be equal to some number k between 0 and n. The number of $2n$-bit strings with exactly k ones in the first half and, consequently, $n-k$ ones in the second half is equal to $\binom{n}{k} \cdot \binom{n}{n-k}$. Summing over all possible values of k gives the expression on the right side of the equation above. Since the two sides of this equation count the elements of the same set in two different ways, the two sides must be equal.

(b)

$$\binom{n+1}{m+1} = \sum_{k=m}^{n} \binom{k}{m}$$

Solution. Let S be the set of $(m+1)$-combinations of the set $\{1, 2, 3, \ldots, n+1\}$. On one hand, $|S| = \binom{n+1}{m+1}$ by the standard formula for combinations. On the other hand, we can select each $(m+1)$-combination by an alternative procedure. First, choose the largest element of the combination, and call this $k+1$. Thus, k must be in the range from m to n. Now we can choose the remaining elements in $\binom{k}{m}$ ways, since we must select m elemens in the range $\{1, 2, 3, \ldots, k\}$. This shows that the expression on the right side of the equation above is also equal to $|S|$. Since both sides are equal to $|S|$, they must be equal to each other.

Problem 5. The "engineering primes" are 2, 3, 5, 7, and all larger natural numbers that are not divisible by 2, 3, 5, or 7.

(a) What is the smallest engineering prime that is not a real prime?

Solution.

$$11 \cdot 11 = 121$$

(b) Use inclusion-exclusion to count the number of engineering primes that are less than a million.

Solution. We can use inclusion-exclusion to count numbers in the range from 1 to million that are multiples of 2, 3, 5, or 7:

$$\begin{aligned}
&\left\lfloor\frac{10^6}{2}\right\rfloor+\left\lfloor\frac{10^6}{3}\right\rfloor+\left\lfloor\frac{10^6}{5}\right\rfloor+\left\lfloor\frac{10^6}{7}\right\rfloor\\
-\left\lfloor\frac{10^6}{2\cdot 3}\right\rfloor-\left\lfloor\frac{10^6}{2\cdot 5}\right\rfloor-\left\lfloor\frac{10^6}{2\cdot 7}\right\rfloor-&\left\lfloor\frac{10^6}{3\cdot 5}\right\rfloor-\left\lfloor\frac{10^6}{3\cdot 7}\right\rfloor-\left\lfloor\frac{10^6}{5\cdot 7}\right\rfloor\\
+\left\lfloor\frac{10^6}{3\cdot 5\cdot 7}\right\rfloor+\left\lfloor\frac{10^6}{2\cdot 5\cdot 7}\right\rfloor+&\left\lfloor\frac{10^6}{2\cdot 3\cdot 7}\right\rfloor+\left\lfloor\frac{10^6}{2\cdot 3\cdot 5}\right\rfloor\\
&-\left\lfloor\frac{10^6}{2\cdot 3\cdot 5\cdot 7}\right\rfloor \quad = \quad 771429
\end{aligned}$$

Therefore, there are $1000000 - 771429 - 228571$ numbers from 1 to a million that are not multiples of 2, 3, 5, or 7. Adding in the numbers 2, 3, 5, and 7, we find that the number of engineering primes less than a million is $228571+4 = 228575$.

Problem 6. A *n-bit boolean function* maps each n-bit string to a single bit; that is, a n-bit boolean function has the form $f : \{0,1\}^n \to \{0,1\}$.

(a) How many n-bit boolean functions are there?

Solution. We can map each of the 2^n elements of the domain to either 0 or 1. Therefore, the number of such functions is:

$$2^{2^n}$$

(b) The n-bit boolean function f *depends on bit* i if there exist n-bit strings b and b' such that b and b' differ only in the i-th bit and $f(b) \neq f(b')$. How many n-bit boolean functions are there that do not depend on the i-th bit?

Solution. Now we can map each of the 2^{n-1} domain elements with a 0 in the i-th position to either 0 or 1. Each of the domain elements with a 1 in the i-th position must be mapped to the same value as the corresponding element with a 0 in the i-th position. Therefore, the number of boolean functions that do not depend on the i-th bit is:

$$2^{2^{n-1}}$$

(c) How many n-bit boolean functions depend on *every* bit?

Solution. Generalizing the argument above, the number of n-bit boolean functions that do not depend on a k specified bits is $2^{2^{n-k}}$. By the inclusion-exclusion formula, the number of functions that depend on every bit is the total number of functions, minus the number that depend on one bit, plus the number the depend on two bits, minus the number that depend on three bits, etc. This gives:

$$\sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \cdot 2^{2^{n-k}}$$

Problem 7. We can obtain a crude variant of Stirling's formula by dropping the $\sqrt{2\pi n}$ term, which leaves:

$$n! \approx \left(\frac{n}{e}\right)^n$$

(a) Use this approximation to estimate the value of:

$$\binom{n}{\alpha n}$$

where α is a number between 0 and 1 such that αn is an integer. Show your work, and put your answer in the form

$$2^{nH(\alpha)}$$

where H is a function of α. (The function H is called the *binary entropy function.* Much like e and π, entropy crops up in all sorts of odd places. Furthermore, entropy plays a central role in the field of information theory.)

Solution.

$$\begin{aligned}
\binom{n}{\alpha n} &= \frac{n!}{(\alpha n)!\ ((1-\alpha)n)!} \\
&\approx \frac{\left(\frac{n}{e}\right)^n}{\left(\frac{\alpha n}{e}\right)^{\alpha n} \cdot \left(\frac{(1-\alpha)n}{e}\right)^{(1-\alpha)n}} \\
&= \frac{1}{\alpha^{\alpha n} \cdot (1-\alpha)^{(1-\alpha)n}} \\
&= 2^{nH(\alpha)}
\end{aligned}$$

where:

$$H(\alpha) = \alpha \log_2\left(\frac{1}{\alpha}\right) + (1-\alpha)\log_2\left(\frac{1}{1-\alpha}\right)$$

(b) Using this formula, approximately how many n-bit strings have exactly $n/3$ ones? (Assume that $n/3$ is an integer.) You answer should be of the form 2^{nc} where c is a constant.

Solution. Stting $\alpha = 1/3$ in the formula above gives:

$$2^{nH(1/3)} \approx 2^{0.918n}$$

(c) Using the same formula, what is the approximate probability of flipping 51% heads in n tosses of a fair coin? (Assume that $0.51n$ is an integer.) Your answer should be of the form 2^{-nc} where c is a constant.

Solution. The probability is:

$$\begin{aligned} \frac{\binom{n}{0.51n}}{2^n} &\approx 2^{n(H(0.51)-1)} \\ &\approx 2^{-0.000289n} \end{aligned}$$

Thus, for example, when $n = 1000000$, the probability of flipping 510000 heads is about 2^{-289}.