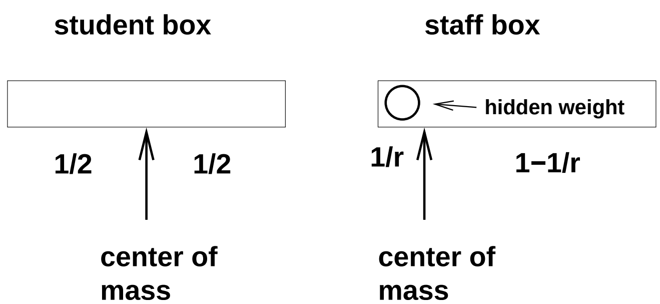
Problem Set 8

Due: Start of class on Tuesday, April 29.

Problem 1. The 6.042 course staff considered a box-stacking competition against a team of students. Naturally, this prompted us to search for the best way to cheat. We had two promising ideas:

(a) Our first idea was to secretly weight all of our boxes, moving the center of mass nearer to one end as shown below:



The students would be given ordinary, unweighted boxes. If both teams built optimal structures, how much farther would the staff's box stack extend? Assume that each team has n boxes.

Solution. As in class, suppose that the edge of the table is position 0. The rightmost edge of the k-th box from the top is at position x_k. The center-of-mass of the top k boxes is at position z_k. Then we reason as follows:

$$\begin{aligned}
x_{k+1} &= z_k \\
&= \frac{z_{k-1} \cdot (k-1) + (x_k - (1 - 1/r)) \cdot 1}{(k-1)+1} \\
&= \frac{x_k \cdot k - (1 - 1/r)}{k} \\
&= x_k - \frac{r-1}{r} \cdot \frac{1}{k}
\end{aligned}$$

This implies that the course staff's stack would extend to $(r-1)/r \cdot H_n$, which is nearly twice as far as the student stack when r is large.

(b) An alternative idea was to use thinner boxes. Students would be given boxes of height 1, but the staff would use boxes of height $1/q$ for some $q > 1$. To create the appearance of fairness, each team would be given a stack of boxes with total height n. Of course, this would give the students n ordinary boxes and would give the staff qn thin boxes. If both teams now built optimal structures, how much farther would the staff's stack extend? Also, find an approximate answer by using the formula $H_k \approx \ln k$.

Solution. Using the analysis from class, the extension of the student stack would be $H_n/2$ and the extension of the staff stack wouuld be $H_{qn}/2$. Thus, the staff's winning margin would be:

$$\begin{aligned} \frac{H_{qn} - Hn}{2} &\approx \frac{\ln qn - \ln n}{2} \\ &= \frac{\ln q}{2} \end{aligned}$$

Thus, for example, if the staff boxes were 1/2-height, our winning margin would be $(\ln 2)/2 = 0.345\ldots$ boxes.

Problem 2. A bug begins walking across a rug. The rug is one meter long, and the bug walks one centimeter per second. However, at the end of each second, the rug is given a tug, which instantaneously stretches it by one meter. The rug stretches uniformly; that is, the portion of the rug behind the bug is stretched by the same factor as the portion ahead of the bug.

To clarify the procedure, here is what happens at the beginning. In the first second, the bug walks 1 centimeter. Then the rug is instantaneously stretched by one meter, leaving 2 cm behind the bug and 198 cm ahead. In the next second, the bug walks another centimeter, leaving 3 cm of rug behind the bug and 197 cm ahead. Then the rug is stretched by another meter, leaving $3 \cdot (3/2) = 4.5$ cm of rug behind the bug and $197 \cdot (3/2) = 295.5$ cm ahead.

Can the bug cross the rug? If so, approximately how long does it take?

Solution. In the k-th second, the rug is $100k$ centimeters long and the bug traverses 1 centimeter. Therefore, the *fraction* of the rug crossed by the bug is $1/100k$. Therefore, the fraction of the rug crossed by the bug in the first n seconds is:

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{100k} &= \frac{1}{100} \cdot H_n \\ &\approx \frac{\ln n}{100} \end{aligned}$$

The bug completes it journey when the fraction of the rug crossed is 1; that is, when:

$$
\begin{aligned}
\frac{\ln n}{100} &\geq 1 \\
\ln n &\geq 100 \\
n &\geq e^{100} \\
&\approx 10^{43}
\end{aligned}
$$

So the bug does cross the rug, but it takes about 10^{43} seconds!

Problem 3. Use integration to find lower and upper bounds on the following infinite sum:

$$
s = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \ldots
$$

To get a really accurate answer, add up the first few terms of the sum explicitly and then bound the remaining terms using integration.

Solution. The sum of the first four terms is:

$$
\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} = \frac{22369}{20736}
$$

We can bound the remaining terms using integrals as follows:

$$
\int_4^\infty \frac{1}{(x+1)^4} \leq \sum_{i=5}^\infty \leq \frac{1}{5^4} + \int_5^\infty \frac{1}{x^4}
$$

On the left, we have:

$$
\begin{aligned}
\int_4^\infty \frac{1}{(x+1)^4} &= -\frac{1}{3(1+x)^3}\bigg|_4^\infty \\
&= \frac{1}{375}
\end{aligned}
$$

On the right, we have:

$$
\begin{aligned}
\frac{1}{5^4} + \int_5^\infty \frac{1}{x^4} &= \frac{1}{5^4} + \left(-\frac{1}{3x^3}\right)\bigg|_5^\infty \\
&= \frac{1}{5^4} + \frac{1}{375}
\end{aligned}
$$

Putting this all together, we have the bounds:

$$\frac{22369}{20736} + \frac{1}{375} \;\leq\; s \;\leq\; \frac{22369}{20736} + \frac{1}{5^4} + \frac{1}{375}$$

In numerical terms, this means:

$$1.08142\ldots \;\leq\; s \;\leq\; 1.08301\ldots$$

The exact sum is $s = \pi^4/90 = 1.0823232337111381915\ldots$.

Problem 4. A very tired bug begins at position $(0,0)$ in the coordinate plane $\mathbb{N} \times \mathbb{N}$. Each second, it either staggers east (incrementing its first coordinate) or wobbles north (incrementing the second coordinate).

(a) By how many different paths can the bug reach point (x, y)?

Solution. The number of distinct path is equal to the number of ways to choose x eastward steps from among a total of $x + y$ steps:

$$\binom{x+y}{x} \;=\; \binom{x+y}{y}$$

(b) At the point (x', y'), where $0 \leq x' \leq x$ and $0 \leq y' \leq y$, there is a Flaming Chasm of Hideous Death. If the bug picks a path to (x, y) uniformly at random, what is the probability that it falls into the Chasm?

Solution. Every path from $(0,0)$ to (x, y) via (x', y'), consists of a path from $(0,0)$ to (x', y') together with a path from (x', y') to (x, y). Using the previous result and the product rule, we find:

$$\Pr\{\text{visits } (x', y')\} \;=\; \frac{\binom{x'+y'}{x'}\binom{(x-x')+(y-y')}{x-x'}}{\binom{x+y}{x}}$$

Problem 5. In the game of seven-card stud, a player is dealt a hand of seven cards. Assume that the deck is thoroughly shuffled, so that every hand is equally likely. Solve the problems below and show your work. You may leave binomial coefficients in your answers.

Note that each card has a *suit* ($\heartsuit$, $\spadesuit$, $\diamondsuit$, or $\clubsuit$) and a *value* (A, 2, 3, 4, …, 9, 10, J, Q, or K). For this problem, the value of an ace is just below the value of a 2. The terms *pair*, *three-of-a-kind*, and *four-of-a-kind* refer to sets of two, three, and four cards with the same value.

(a) How many different hands are possible?

Solution. There is one hand for each way of choosing 7 cards from a 52-card deck. Therefore, the number of possible hands is:

$$\binom{52}{7}$$

(b) What is the probability that all cards in a hand have the same suit?

Solution. The suit can be chosen in 4 ways. The values of the seven cards can be chosen in $\binom{13}{7}$ ways. Therefore, the probability is:

$$\Pr\{\text{all same suit}\} = \frac{4 \cdot \binom{13}{7}}{\binom{52}{7}}$$

(c) What is the probability that a hand contains seven cards with the same suit and consecutive values?

Solution. There are 7 ways to chose the value of the lowest card, and there are 4 ways to chose the suit. Therefore, the probability is:

$$\Pr\{\text{7-card straight flush}\} = \frac{7 \cdot 4}{\binom{52}{7}}$$

(d) What is the probability that a hand contains a four-of-a-kind?

Solution. The value of the four-of-a-kind can be chosen in 13 ways. The rest of the hand may consist of any three cards chosen from the remaining 48, giving $\binom{48}{3}$ possibilities. Overall, the probability is:

$$\Pr\{\text{4-of-a-kind}\} = \frac{13 \cdot \binom{48}{3}}{\binom{52}{7}}$$

(e) What is the probability that a hand contains exactly one pair and no three- or four-of-a-kind?

Solution. First, we can choose the value of the pair in 13 ways. Then we must select two of the fours cards with that value; this can be done in $\binom{4}{2}$ ways. The other five cards must have distinct values drawn from the remaining 12 possible values. This can be done in $\binom{12}{5}$ ways. For each of these five cards, we can chose the suit in 4 ways, giving another 4^5 possibilities. In all, we have:

$$\Pr\{\text{one pair}\} = \frac{13 \cdot \binom{4}{2} \cdot \binom{12}{5} \cdot 4^5}{\binom{52}{7}}$$

(f) What is the probability that a hand contains two three-of-a-kinds, but no four-of-a-kind?

Solution. The values of the three of a kinds can be chosen in $\binom{13}{2}$ ways. The suits represented in each three of a kind can be chosen in $\binom{4}{3}$ ways. The remaining card can be chosen from among the $52 - 8 = 44$ with a different value. In total, we have:

$$\Pr\{\text{two three-of-a-kind}\} = \frac{\binom{13}{2} \cdot \binom{4}{3}^2 \cdot 44}{\binom{52}{7}}$$

(g) What is the probability that there is no pair, no three-of-a-kind, and no four-of-a-kind?

Solution. This is equivalent to asking for the probability that all cards have different values. In that case, the values can be chosen in $\binom{13}{7}$ ways, and the suit of each card can be chosen in four ways. Therefore, the probability is:

$$\Pr\{\text{no multiples}\} = \frac{\binom{13}{7} \cdot 4^7}{\binom{52}{7}}$$

Problem 6. There are 100 soldiers. Each morning, they line up for review in 10 rows with 10 soldiers in each row. This continues for 60 days. Prove that there must exist two soldiers who lined up beside each other in the same row on at least two different occasions.

Solution. We use the pigeonhole principle. The holes are all the pairs of soldiers $\{x, y\}$. A pigeon is a pair $(d, \{x, y\})$ where x and y are soldiers that stand next to each other on day number d. The pigeon $(d, \{x, y\})$ is mapped to the hole $\{x, y\}$.

The number of holes is $\binom{100}{2} = 4950$. The number of pigeons is $60 \cdot 10 \cdot 9 = 5400$. Since there are more pigeons than holes, two pigeons must be mapped to the same hole. That is, there must be two soldiers that lined up next to each other on at least two different occasions.

Problem 7. Prove that the following equation holds for all $r, n \in \mathbb{N}$:

$$\binom{r}{0} + \binom{r+1}{1} + \binom{r+2}{2} + \cdots + \binom{r+n}{n} = \binom{r+n+1}{n}$$

Solution. The proof is by induction on n. The induction hypothesis is the equation above. For the base case, $n = 0$, this equation becomes:

$$\binom{r}{0} = \binom{r+1}{0}$$

This always holds, because both binomials are equal to 1. Now suppose that the equation holds for some $n \in \mathbb{N}$. We show that it holds for $n+1$ also as follows:

$$\begin{aligned} \binom{r}{0} + \binom{r+1}{1} + \cdots + \binom{r+n}{n} + \binom{r+n+1}{n+1} &= \binom{r+n+1}{n} + \binom{r+n+1}{n+1} \\ &= \binom{r+(n+1)+1}{n+1} \end{aligned}$$

The first step uses the induction hypothesis, and the second uses the standard identity:

$$\binom{x}{y} = \binom{x-1}{y} + \binom{x-1}{y-1}$$

The claim follows by induction.