Problem Set 6

Due: Start of class on Tuesday, April 8.

Problem 1. The following equation holds for all $n \in \mathbb{N}$.

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) \quad = \quad \frac{n(n+1)(n+2)(n+3)}{4}$$

(a) Prove this fact using induction.

Solution. The proof is by induction. Let $P(n)$ be the proposition that:

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) \quad = \quad \frac{n(n+1)(n+2)(n+3)}{4}$$

First, note that $P(0)$ holds, because both sides of this equation are equal to zero. Now we must show that for all $n \geq 0$, $P(n)$ implies $P(n+1)$. Suppose that $P(n)$ is true, where $n \geq 0$. Then we can reason as follows:

$$\begin{aligned}
&1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) + (n+1)(n+2)(n+3) \\
&\quad = \frac{n(n+1)(n+2)(n+3)}{4} + (n+1)(n+2)(n+3) \\
&\quad = \frac{(n+1)(n+2)(n+3)(n+4)}{4}
\end{aligned}$$

This shows that $P(n)$ implies $P(n+1)$. Therefore, the claim holds by induction.

(b) Prove this fact using well-ordering.

Solution. The proof uses well-ordering. Suppose that the claim is false. Let S be the set of all $n \in \mathbb{N}$ such that the equation does not hold. By our supposition, S is nonempty, and by the well-ordering priniciple, S contains a smallest element. This smallest element cannot be 0; in the case $n = 0$, both sides of the equation are equal to zero and are therefore equal to each other. Therefore, the smallest element of S must be some integer $s > 0$. Since $s - 1$ is in $\mathbb{N}$ and is not in S, the equation must hold for $n = s - 1$. Thus we have:

$$\begin{aligned}
1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + (s-1)s(s+1) + s(s+1)(s+2) \\
&= \frac{(s-1)s(s+1)(s+2)}{4} + s(s+1)(s+2) \\
&= \frac{s(s+1)(s+2)(s+3)}{4}
\end{aligned}$$

The first step uses the fact that the equation holds for $n = s - 1$, and the second step uses only algebra. This shows that the equation holds for $n = s$, contradicting the definition of s. Therefore, our original supposition was wrong, and the claim is true.

Problem 2. Is the proof of the following claim valid? If not, identify the first erroneous statement and explain why it is wrong.

Claim *Every nonegative rational number is equal to a^2/b^2 for some pair of nonegative integers a and b.*

Proof. The proof uses well-ordering. Suppose that the claim is false; that is, there is at least one nonnegative rational number that cannot be written as a^2/b^2 for some pair of nonnegative integers a and b. Let q be the smallest such number. Then, since $q' = q/4$ is a nonegative rational number smaller than q, there exist nonnegative integers a and b such that $q' = a^2/b^2$. But then q is also equal to a ratio of squares, namely $(2a)^2/b^2$. This contradicts the definition of q. Therefore, our original supposition must be false, and so the claim is true. ∎

Solution. The proof is not valid. The first erroneous phrase is "Let q be the smallest such number." There is no assurance that such a q exists. The well-ordering principle guarantees that every subset of $\mathbb{N}$ has a smallest element, but it does not guarantee that every subset of the positive rational numbers has a smallest element.

In fact, the set of counterexamples to the claim has no smallest element; every term in the following, strictly-decreasing sequence is a counterexample:

$$\frac{1}{2}, \quad \frac{1}{2^3}, \quad \frac{1}{2^5}, \quad \frac{1}{2^7}, \quad \ldots$$

In other words, we are claiming that $1/2^{2k+1}$ is not a ratio of perfect squares for any $k \in \mathbb{N}$. Suppose this were not the case; that is, there existed a and b such that:

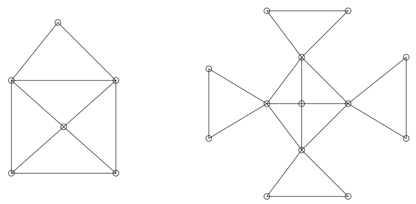
$$\frac{1}{2^{2k+1}} = \frac{a^2}{b^2}$$

Then it follows that:

$$b^2 = a^2 \cdot 2^{2k+1}$$

But this is impossible; 2 divides the left side an even number of times, but divides the right side an odd number of times.

Problem 3. Try drawing the figures below without lifting your pencil or running over the same segment twice.



You should be able to draw the first, but not the second. Note that these figures can be interpreted as graphs. The little circles represent vertices and the lines between them indicate edges. Call a graph that can be drawn without lifting your pencil or retracing a line *drawable*.

Now consider the following two conditions on a graph:

1. The graph is connected; that is, there is a path between every pair of vertices.
2. The number of vertices with odd degree is either zero or two.

(a) Prove that every drawable graph satisfies both of these conditions.

Solution. To solve this problem, we need a precise idea of what a path is. A path is a sequence $(a_1, \cdots, a_n)$ of vertices, such that $a_i \neq a_{i+1}$. If the sequence contains n vertices, then we will say that the path has a length of $n-1$. Therefore, we will conventionally consider that the empty path has a length of -1. A path is closed if $a_1 = a_n$, it is open otherwise. A graph is drawable if there is a path $(a_1, \cdots, a_n)$ such that each node is a_i for some i, and for each vertex, there is exactly one i such that $\{a_i, a_{i+1}\}$ is that vertex (note that $\{u, v\} = \{v, u\}$).

Property 1 is easy to show. Indeed, if a graph is drawable, then there exists a path p that goes through every vertex and edge of the graph without going through an edge twice. For any pair (a, b) of vertices in the graph, those vertices must therefore appear at least once each in p. Consider the subpath of p that lies between some instance of a and some instance of b. That subpath goes from a to b, or b to a. The existence of such a subpath for any pair of vertices in the graph implies that the graph is connected.

The empty graph trivially satisfies Property 2. For other drawable graphs, the path needed to draw the graph has length at least 0. We shall show property 2 by induction on that length. Let $P(n)$ be the predicate "a drawable graph that can be drawn in n edges is of one of two types: either it only has even degree vertices, and it is drawn by a closed path; either it has exactly two odd degree vertices, and can be drawn by an open path".

- Base case: The only graphs that can be drawn with a path of length of 0 is the single-node graph. Its single node has an even number of edges, and the path (v) used to draw it starts and ends on the same vertex. So $P(0)$ holds.
- Inductive step: Assume that $P(n)$ holds for $n > 0$. Consider any drawable graph G that can be drawn by a path p of length $n+1$. Let s and e be the first and last vertices in p. Let q be the path that results from removing the last virtex from p, it has a length of n. Let H be the graph that is drawn by q. By definition, H is drawable, therefore it satisfise the conditions laid out in $P(n)$.
  - Case 1: If H only has even-degree vertices, then let s be the last vertex of q (the first and last elements of q are the same by the induction hypothesis). In H, s is an even degree edge, and if e is present it is also an even degree edge. Because they are consecutive in p, s and e are distinct, so p is an open path. Moreover, by adding the edge $\{s, e\}$ to H to make G, s becomes an odd-degree vertex, and whether it was present in H or not, e becomes an odd-degree vertex. All the other vertices in H are unchanged by the operation and remain even-degree edges. Therefore, G is a graph with exactly two odd degree vertices, and the odd degree vertices are s and e which are the first and last vertices of the open path p which defines G.
  - Case 2: If H has exactly two odd-degree vertices, then let a be the last vertex in q. By the induction hypothesis s and a are odd degree vertices in H. When we add $\{a, e\}$ to H to make it into G, a becomes an even degree vertex since a and e, being consecutive in p, must be distinct.
    * Case 2.1: If s and e are distinct vertices, then when going from H to G, s remains an odd degree vertex, while e becomes an odd degree vertex (it was even degree or absent in H being distinct from s and a). All unmentioned vertices preserve their parity from H to G. Therefore, there are exactly 2 odd degree vertices s and e in G, and the path that defines G is open since s and e are distinct.
    * Case 2.2: If $s = e$ then when going from H to G, s becomes an even degree vertex. All unmentioned vertices preserve their parity from H to G. Therefore G only has even degree vertices, and is defined by a closed path.

  Thus in each case we have seen that the requirements for P(n+1) are met, so P(n+1) holds.

We may conclude by induction on n that $P(n)$ holds for all n. Thus for any drawable graph Property 2 holds, either because the graph is empty, or because of $P(n)$ where n is the number of edges in the graph.

(b) Prove that every graph satisfying both of these conditions is drawable.

Solution.

Will be provided later.

Problem 4. How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that some pair adds up to 16? Justify your answer using the pigeonhole principle; in particular, what are the pigeons and what are the holes?

Solution. We shall show using pigeonhole principle that we need to pick five elements to be sure that two of them add up to 16.

We have 4 holes: A, B, C and D. The pigeons are the elements of $S = \{1, 3, 5, 7, 9, 11, 13, 15\}$. Given a subset of S, we place 1 and 15 in A, 3 and 13 in B, 5 and 11 in C, and 7 and 9 in D. By the pigeonhole principle, if we pick a subset of S with at least 5 elements, then two (distinct) elements will end up in the same hole. Because of the way the pigeons have been assigned to holes, those two elements must add up to 16. Therefore, if we pick 5 elements, we are sure that two of them add up to 15.

Noting that $\{1, 3, 5, 7\}$ is a 4 element subset of S that does not have a pair of elements that add up to 16, we can conclude that 5 is the smallest number of elements we need to pick to be sure to find a pair that adds up to 16.

Problem 5. There is a group of people who were all born in 1983. Below is a list of properties that the group may or may not possess. For each property, either give the minimum number of people that must be in the group to ensure that the property holds, or else indicate that the property need not hold even if the group is arbitrarily large. No proofs are required.

(a) At least 2 people were born on the same day.

Solution. This property holds for every group of 366 or more people. With only 365 people, the people could all be born on different days. But by the pigeonhole principle, where the 365 days of the year are holes, and the people are pigeons, the property must hold for 366 people or more.

Note that there are two parts in this proof. First we see that 365 is not enough, then that 366 is sufficient. If we only applied pigeonhole, we would know that 366 is sufficient, but we wouldn't sure that 365 isn't sufficient.

(b) At least 2 people were born on January 1.

Solution. This property need not hold, even for arbitrarily large groups of people. Indeed, they may all be born on April 1^{st}, for example.

(c) At least 3 people were born on the same day of the week.

Solution. This property holds for every group of 15 or more people. With 14 people, each day of the week can correspond to two birthdays, but with 15, we can apply the general pigeonhole principle with days of the week as holes and people as pigeons.

(d) At least 4 people were born in the same month.

Solution. This property holds for every group of 37 or more people. With 36 people, three can be born on each month, but with 37, we can apply the general pigeonhole principle with months as holes and people as pigeons.

(e) At least 2 people were born exactly one week apart.

Solution. This property need not hold. For example, everyone might be born on the same day of the year.

(f) At least 2 people were born on the same day or else 2 people were born exactly one week apart.

Solution. 184 people. Indeed, 183 people can be accomodated without satisfying the property by having them born on alternate days of the year (Jan 1, Jan 3, ... , Dec 29, Dec 31). Let us see pairs of days, one of which is in 1983, and which are one week apart, as holes. There are $365 + 7 = 372$ holes: one from each day in 1983 to the day one week before, and one from each of the seven first days of 1984 to the day in 1983 that was one week before. Each person counts as two pigeons, mobilizing the two holes corresponding to the person's birthday. For 373 pigeons, we are sure that some pair will be picked twice. For that we need $\lfloor 373/2 \rfloor = 184$ people.

(g) At least 9 people were born on a weekend or at least 11 people were born during the week.

Solution. This property holds for every group of at least $(9-1)+(11-1)+1 = 19$ or more people. For 18 people, 9 can be born on the weekend and 11 on a weekday. But if there are 19 people, either 9 or more are born on the weekend, or at most 8 are born on a week end, which implies that $19-8 = 11$ or more are born during the week. Therefore, the property must hold for 19 people.

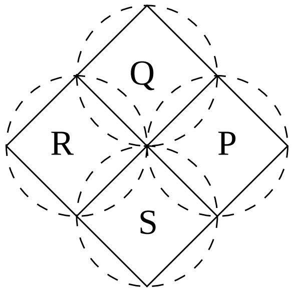
(h) At least n people were born on the n-th day of the year for some n, where days are numbered 1, 2, ... , 365.

Solution. This property holds for at least $1+\sum_{i=0}^{364} i = 364*365/2+1 = 66431$ people. Indeed, if a_i is the number of people born on the i^{th} day, the total number of people is $n = \sum_{i=1}^{365} a_i$. If $a_i = i - 1$ then the property doesn't hold, and $n = \sum_{i=1}^{365}(i-1) = 66430$, so the property doesn't hold for that many people. For the property not to hold, we must have for each i, $a_i <= i - 1$, therefore

$n <= \sum_{i=1}^{365}(i-1) = 66430$. By the contrapositive, for 66431 people or more, the property must hold.

Problem 6. Let D be the set of all points (x, y) in the real plane such that $|x| + |y| \leq 1$. Prove that among every set of five points in D, there exist two points whose distance from one another is at most 1.

Solution. We solve this problem using the pigeonhole principle. A little geometry shows that D is in fact a square of edge size $\sqrt{2}$, which can be tiled by four smaller squares of edge size $\sqrt{2}/2$. The diagonal of each smaller square is therefore 1, so each one is inscribed in a circle of diameter 1. The whole construction is shown here:



We shall take the four small squares to be our holes, and the five points we wish to place in D to be our pigeons. Since the four small squares cover D, each pigeon must fall into a hole (it might actually fall on more than one because adjacent squares share an edge). By the pigeonhole principle, two points must be in the same square. They are therefore in the same circle of diameter 1, and thus must be within 1 of each other.

Problem 7. There is a meeting of n people. Each pair of people at the meeting either shakes hands or does not shake hands. Prove that there must exist two people at the meeting who shook the same number of hands.

Solution. We solve this problem using the pigeonhole principle. Let the pigeons be the n people at the meeting, and the holes be number of times times that a person has shaken hands.

First we consider the case where some person a hasn't shaken any hands. In that case the most hands any other person can shake is $n-2$ (people don't shake hands with themselves or with a). So the number of hands a person can have shaken is between 0 and $n-2$. Thus there are $n-1$ possible values for the number of handshakes a person has exchanged. By the pigeonhole principle, at least two of the n people must have shaken the same number of hands.

We now consider the complementary case where everybody has exchanged at least one handshake. Now the number of handshakes for a person is between 1 and $n-1$. Thus there are still $n-1$ possible values for the number of handshakes, and the pigeonhole principle allows to conclude that two people have shaken hands the same number of times.

We can conclude that in any case, two people must have shaken the same number of hands.

Problem 8. Suppose that during each day of an 13-week term, an MIT student feels overwhelmed at least once. (Unfortunately, this is going to be one of those times.) However, suppose that the student never feels overwhelmed more than 12 times in any one of the 13 weeks. Prove that there exists a sequence of consecutive days during which the student feels overwhelmed exactly 25 times.

Solution. Let P_i be the total number of times the student feels overwhelmed in the first i days. Since the student feels overwhelmed at least once every day, the sequence $P_0, P_2, \ldots, P_{91}$ is strictly increasing. Since the student is overwhelmed at most 12 times per week, the final term, P_{91}, is at most $13 \cdot 12 = 156$. Now the sequence $P_0+25, P_1+25, \ldots, P_{91}+25$ is also strictly increasing, and the largest term, $P_{91}+25$, is at most $156+25 = 181$. Therefore, the 184 values $P_0, \ldots, P_{91}$, $P_0 + 25, \ldots, P_{91} + 25$ all lie between 0 and 181. By the pigeonhole principle, at least two of these values must be the same. Since no two of the numbers $P_1, \ldots, P_{91}$ are equal, and no two of the numbers $P_1 + 25, \ldots, P_{91} + 25$ are equal, there must be an i and a j such that $P_i = P_j + 25$. This implies that on days $j+1, j+2, \ldots, i$, the student is overwhelmed exactly 25 times.