Problem Set 2

Due: Start of class on Tuesday, February 25.

Problem 1. Suppose that *Let's Make a Deal* is played according to different rules. Now there are four doors, with a prize hidden behind one of them. The contestant is allowed to pick a door. Carol must then reveal a different door that has no prize behind it. The contestant is allowed to stay with his or her original door or to pick one of the other two that are still closed. If the contestant chooses the door concealing the prize in this second stage, then he or she wins.

(a) Contestant Stu, a sanitation engineer from Trenton, New Jersey, stays with his original door. What is the probability that he wins the prize?

Solution. Stu guesses the right door initially with probability $\frac{1}{4}$ and guesses the wrong door with probability $\frac{3}{4}$. If he was right initially, then he surely wins the prize. If he was wrong initially, then he surely does not win the prize. Therefore, his probability of winning is $\frac{1}{4} \cdot 1 + \frac{3}{4} \cdot 0 = \frac{1}{4}$.

(b) Contestant Zelda, an alien abduction researcher from Helena, Montana, randomly picks one of the other two doors with equal probability. What is the probability that she wins the prize?

Solution. Zelda also guesses the right door initially with probability $\frac{1}{4}$ and guesses the wrong door with probability $\frac{3}{4}$. If she was right initially, then she can not win. If she was wrong initially, then she wins with probability $\frac{1}{2}$. Therefore, her probability of winning is $\frac{1}{4} \cdot 0 + \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$.

Problem 2. Inspector Lestrade is waiting for Holmes and Watson, who are driving (separately) to meet him. It is a winter day, which means that the roads have a 70% chance of being icy. Lestrade's secretary informs him that Watson has had a car accident. Lestrade figures that each driver has an 80% chance of an accident if the roads are icy, and a 10% chance of an accident if they are not. Given that the roads are icy, the event that Watson has an accident is independent of the event that Holmes has an accident. Given that the roads are not icy, these events are again independent.

(a) "Probably the roads are icy", Lestrade says. What is the actual probability that the roads are icy, given that Watson had an accident?

Solution. In this question, we shall find that knowing that Watson has had an accident, the probability that the roads are icy is about 94.9%. We could solve this problem in two ways: using a tree diagram, or using conditional probability formulas. We shall use the conditional probability method, for variety.

Let I be the event "the road is icy", and W be the event "Watson has an accident". In this question, we want to know the probability that the road is icy, knowing that Watson had an accident. By applying the definition of conditional probability twice, we get

$$\begin{aligned}\Pr\{I \mid W\} &= \frac{\Pr\{I \cap W\}}{\Pr\{W\}} \\ &= \frac{\Pr\{W \mid I\} \cdot \Pr\{I\}}{\Pr\{W\}}\end{aligned}$$

In this expression, we know that $\Pr\{W \mid I\} = 0.8$ and $\Pr\{I\} = 0.7$. We still need to compute $\Pr\{W\}$. We do so using the total probability law from Problem 7:

$$\begin{aligned}\Pr\{W\} &= \Pr\{W \mid I\} \cdot \Pr\{I\} + \Pr\left\{W \mid \overline{I}\right\} \cdot \Pr\left\{\overline{I}\right\} \\ &= 0.8 \cdot 0.7 + 0.1 \cdot 0.3 \\ &= 0.59\end{aligned}$$

Substituting this into our previous expression, we conclude:

$$\begin{aligned}\Pr\{I \mid W\} &= \frac{\Pr\{W \mid I\} \cdot \Pr\{I\}}{\Pr\{W\}} \\ &= \frac{0.8 \cdot 0.7}{0.59} \\ &= 0.949\ldots\end{aligned}$$

Therefore, there is about a 94.9% chance that the roads are icy, knowing that Watson had an accident.

(b) "Therefore, Holmes will probably crash as well, and so I can go to lunch", he reasons. What is the probability that Holmes has an accident, given that Watson did?

Solution. Let H be the event "Holmes has an accident", and define events W and I as before.

$$
\begin{aligned}
\Pr\{H \mid W\} &= \frac{\Pr\{H \cap W\}}{\Pr\{W\}} \\
&= \frac{\Pr\{H \cap W \mid I\}\Pr\{I\} + \Pr\{H \cap W \mid \overline{I}\}\Pr\{\overline{I}\}}{\Pr\{W\}} \\
&= \frac{\Pr\{H \mid I\}\Pr\{W \mid I\}\Pr\{I\} + \Pr\{H \mid \overline{I}\}\Pr\{W \mid \overline{I}\}\Pr\{\overline{I}\}}{\Pr\{W\}} \\
&= \frac{0.8 \cdot 0.8 \cdot 0.7 + 0.1 \cdot 0.1 \cdot 0.3}{0.59} \\
&= 0.764\ldots
\end{aligned}
$$

The first step uses the definition of conditional expectation, and the second usees the total probability law from Problem 7. The third step uses the independence assumptions. All that remains is to plug in values from the problem statement and the previous problem part. We conclude that if Watson has had an accident, then the probability of Holmes having an accident is about 76.4%.

(c) "If you look out your window, Inspector, you'll find that the roads are not icy at all", says his secretary. What is the probability that Holmes has an accident, given that Watson did, but the roads are not icy?

Solution. When the roads are not icy, the events of Holmes and Watson having accidents are independent. Therefore the probability of Holmes having an accident is simply the probability that he has an accident knowing that it not icy, which is 10%.

We can prove this formally as follows:

$$
\begin{aligned}
\Pr\{H \mid \overline{I} \cap W\} &= \frac{\Pr\{H \cap \overline{I} \cap W\}}{\Pr\{\overline{I} \cap W\}} \\
&= \frac{\Pr\{H \cap W \mid \overline{I}\} \cdot \Pr\{\overline{I}\}}{\Pr\{W \mid \overline{I}\} \cdot \Pr\{\overline{I}\}} \\
&= \frac{\Pr\{H \cap W \mid \overline{I}\}}{\Pr\{W \mid \overline{I}\}} \\
&= \frac{\Pr\{H \mid \overline{I}\} \cdot \Pr\{W \mid \overline{I}\}}{\Pr\{W \mid \overline{I}\}} \\
&= \Pr\{H \mid \overline{I}\}
\end{aligned}
$$

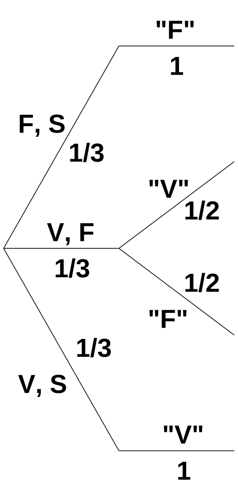
As expected, the probability of Holmes crashing, knowing that it is not icy and that Watson has crashed, is 10%.

Problem 3. There are three prisoners in a maximum-security penitentiary for fictional villains: the Evil Wizard Voldemort, the Dark Lord Sauron, and Little Bunny Foo-Foo. The parole board has declared that it will release two of the three, chosen uniformly at random, but has not yet released their names. Naturally, Sauron figures that he will be released to his home in Mordor, where the shadows lie, with probability $\frac{2}{3}$.

A guard offers to tell Sauron the name of one of the other prisoners who will be released (either Voldemort or Foo-Foo). However, Sauron declines the offer. He reasons that if the guard says, for example, "Little Bunny Foo-Foo will be released", then his own probability of release will drop to $\frac{1}{2}$. This is because he will then know that either he or Voldemort will also be released, and these two events are equally likely.

Using conditional probability, either prove that the Dark Lord Sauron has reasoned correctly or prove that he is wrong. Assume that if the guard has a choice of naming either Voldemort or Foo-Foo (because both are to be released), then he names one of the two uniformly at random.

Solution. Sauron has reasoned incorrectly. His mistake is not realizing that the event "Foo-Foo will be released" is different from the event "The guard says Foo-Foo will be released". The tree diagram below shows that those two events are different sets of outcomes. The first split in the tree is based on which two of the three prisoners are to be released, and the second is based on what the guard tells Sauron.



Probability	Guard: "Foo−Foo"	Foo−Foo released	Sauron released
1/3	yes	yes	yes
1/6	no	yes	no
1/6	yes	yes	no
1/3	no	no	yes

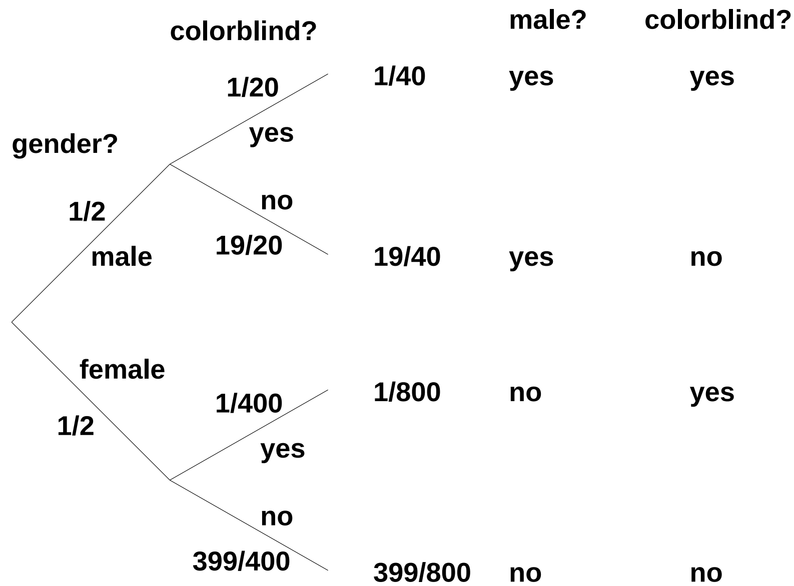
Taking probabilities from the tree diagram, we can compute Sauron's true probabiliy of release as follows:

$$
\begin{aligned}
&\Pr\{\text{Sauron released} \mid \text{guard says "Foo-Foo"}\} \\
&\qquad = \frac{\Pr\{\text{Sauron released} \cap \text{guard says "Foo-Foo"}\}}{\Pr\{\text{guard says "Foo-Foo"}\}} \\
&\qquad = \frac{1/3}{1/3 + 1/6} \\
&\qquad = \frac{2}{3}
\end{aligned}
$$

Therefore, if the guard says that Little Bunny Foo-Foo will be released, then Sauron, Lord of Mordor, still has a $\frac{2}{3}$ chance of release.

Problem 4. Suppose that, on average, 5 men out of 100 and 25 women out of 10,000 are colorblind. A colorblind person is chosen uniformly at random from a population with equal numbers of males and females. What is the probability of that person being male?

Solution. We shall solve this problem using the tree diagram below:



Let M be the event "a person is a man", and let C be the event "a person is colorblind". Working from the tree diagram, we can compute the probability that a colorblind person is male as follows:

$$\begin{aligned}
\Pr\{M \mid C\} &= \frac{\Pr\{M \cap C\}}{\Pr\{C\}} \\
&= \frac{1/40}{1/40 + 1/800} \\
&= \frac{20}{21}
\end{aligned}$$

Problem 5. You flip a penny, a dime, a nickel, and a quarter. Assume that the coins are perfectly fair and the outcome of one is unaffected by the outcomes of the others.

(a) What is the probability that you flip at least two heads, given that some coin comes up heads?

Solution. iThe sample space for this experiment consists of sixteen equally-lkely outcomes:

$$\{H, T\}^4 = \{HHHH, HHHT, HHTH, HHTT, HTHH, \ldots\}$$

The four symbols in each sequence specify what comes up on the penny, dime, nickel, and quarter. We can compute the desired probability as follows:

$$\begin{aligned}
&\Pr\{\text{at least two heads} \mid \text{at least one head}\} \\
&\qquad = \frac{\Pr\{\text{at least two heads} \cap \text{at least one head}\}}{\Pr\{\text{at least one head}\}} \\
&\qquad = \frac{\Pr\{\text{at least two heads}\}}{\Pr\{\text{at least one head}\}}
\end{aligned}$$

There is at least one head in all but one of the sixteen outcomes ($TTTT$), and there are at least two heads in all but five of the outcomes ($TTTT$, $TTTH$, $TTHT$, $THTT$, and $HTTT$). Therefore, we have:

$$\begin{aligned}
&\Pr\{\text{at least two heads} \mid \text{at least one head}\} \\
&\qquad = \frac{\Pr\{\text{at least two heads}\}}{\Pr\{\text{at least one head}\}} \\
&\qquad = \frac{11/16}{15/16} \\
&\qquad = \frac{11}{15}
\end{aligned}$$

The probability of getting at least two heads, knowing that there is at least one head is $\frac{11}{15} \approx 73.3\%$.

(b) What is the probability that you flip at least two heads, given that the dime comes up heads?

Solution. As before, we begin with the definition of conditional probability:

$$\begin{aligned} & \Pr\{\text{at least two heads} \mid \text{dime is heads}\} \\ & \quad = \frac{\Pr\{\text{at least two heads} \cap \text{dime is heads}\}}{\Pr\{\text{dime is heads}\}} \end{aligned}$$

In 8 out of 16 outcomes, the dime comes up heads. In 7 of those outcomes, there are at least two heads overall; the only exception is $TTHT$. Therefore, we have:

$$\begin{aligned} \Pr\{\text{dime is heads}\} &= \frac{8}{16} \\ \Pr\{\text{at least two heads} \cap \text{dime is heads}\} &= \frac{7}{16} \end{aligned}$$

By plugging these numbers into the formula above, we conclude that:

$$\begin{aligned} & \Pr\{\text{at least two heads} \mid \text{dime is heads}\} \\ & \quad = \frac{\Pr\{\text{at least two heads} \cap \text{dime is heads}\}}{\Pr\{\text{dime is heads}\}} \\ & \quad = \frac{7/16}{8/16} \\ & \quad = \frac{7}{8} \end{aligned}$$

So the odds of at least two coins being heads, knowing that the dime is heads, are $\frac{7}{8} = 87.5\%$.

Problem 6. You have two decks of cards. One of them is complete, but one of them is missing the ace of spades. You do not remember which is which. You pick a deck at random and pull out a card uniformly at random. If that card is the seven of hearts, what is the probability that you are holding the complete deck?

Solution. Let C be the event that you are holding the complete deck, and let H be the event that you draw the seven of hearts. We want to compute:

$$\begin{aligned}
\Pr\{C \mid H\} &= \frac{\Pr\{C \cap H\}}{\Pr\{H\}} \\
&= \frac{\Pr\{H \mid C\} \cdot \Pr\{C\}}{\Pr\{H \mid C\} \cdot \Pr\{C\} + \Pr\{H \mid \overline{C}\} \cdot \Pr\{\overline{C}\}}
\end{aligned}$$

The probability of drawing the seven of hearts is $1/52$, given that you selected the complete deck, and $1/51$, given that you selected the incomplete one. Since a deck was selected at random, we have $\Pr\{C\} = \Pr\{\overline{C}\} = 1/2$. Plugging these numbers into the formula above, we find:

$$\begin{aligned}
\Pr\{C \mid H\} &= \frac{1/52 \cdot 1/2}{1/52 \cdot 1/2 + 1/51 \cdot 1/2} \\
&= \frac{51}{103}
\end{aligned}$$

When we pull a seven of hearts from a deck chosen at random, the probability that the deck is complete is about 49.5%. As we would intuitively expect, this is just under 50%.

Problem 7. The *Total Probability Law* is often helpful in evaluating the probability of an event:

$$\Pr\{A\} = \Pr\{A \mid B\} \cdot \Pr\{B\} + \Pr\{A \mid \overline{B}\} \cdot \Pr\{\overline{B}\}$$

Intuitively, this formula lets you break evaluation of the probability of event A into two cases: when event B occurs and when it does not.

(a) Prove that this formula holds for arbitrary events A and B over a finite sample space.

Solution. Note that each outcome in the event A is also either in the event B or in the event $\overline{B}$. Therefore, we can reason as follows:

$$\begin{aligned}
\Pr\{A\} &= \Pr\{A \cap B\} + \Pr\{A \cap \overline{B}\} \\
&= \Pr\{A \mid B\} \cdot \Pr\{B\} + \Pr\{A \mid \overline{B}\} \cdot \Pr\{\overline{B}\}
\end{aligned}$$

The second step uses the definition of conditional probability. Note that the second step is valid only if $0 < \Pr\{B\} < 1$, a condition we inadvertantly omitted from the problem statement.

Our mistake raises an interesting point, however. In many ways, one is better off simply adopting the convention that $\Pr\{A \mid B\} \cdot \Pr\{B\} = 0$ when $\Pr\{B\} = 0$. Then the equation

$$\Pr\{A \cap B\} \;=\; \Pr\{A \mid B\} \cdot \Pr\{B\}$$

holds unconditionally. A similar "undefined quantity times zero equals zero" convention is used routinely in the field of *information theory*. There, by convention, $x \log x$ is treated as 0 when $x = 0$.

(b) Suppose that you flip a fair coin. If you get heads, then you roll a six-sided die and take the result. If you get tails, then you roll two six-sided dice and take the sum as your result. Overall, what is the probability that the result is 6?

Solution. This exercise is a simple application of the total probability law. If 6 is the event "The result is 6" and H is the event "The coin came up heads" then we want to compute

$$\Pr\{6\} \;=\; \Pr\{6 \mid H\} \cdot \Pr\{H\} + \Pr\left\{6 \mid \overline{H}\right\} \cdot \Pr\left\{\overline{H}\right\}$$

Most of these probabilities are given in the problem statement: $\Pr\{H\} = \Pr\{\overline{H}\} = \frac{1}{2}$ and $\Pr\{6 \mid H\} = \frac{1}{6}$. The probability of getting a result of 6 when two dice are rolled is slightly more involved as we must consider the different ways that two dice can add up to 6. We shall simply enumerate them: 1+5, 2+4, 3+3, 4+2, 5+1. So there are 5 ways for the sum of two dice to add up to 6, out of a total of $6 \times 6 = 36$ equally-probable outcomes. Therefore $\Pr\{6 \mid \overline{H}\} = \frac{5}{36}$. We can now compute the answer:

$$\begin{aligned}
\Pr\{6\} &= \Pr\{6 \mid H\} \cdot \Pr\{H\} + \Pr\left\{6 \mid \overline{H}\right\} \cdot \Pr\left\{\overline{H}\right\} \\
&= \frac{1}{6} \cdot \frac{1}{2} + \frac{5}{36} \cdot \frac{1}{2} \\
&= \frac{11}{72}
\end{aligned}$$

(c) Suppose that you carry out the experiment above and the result is 6. What is the probability that you flipped heads?

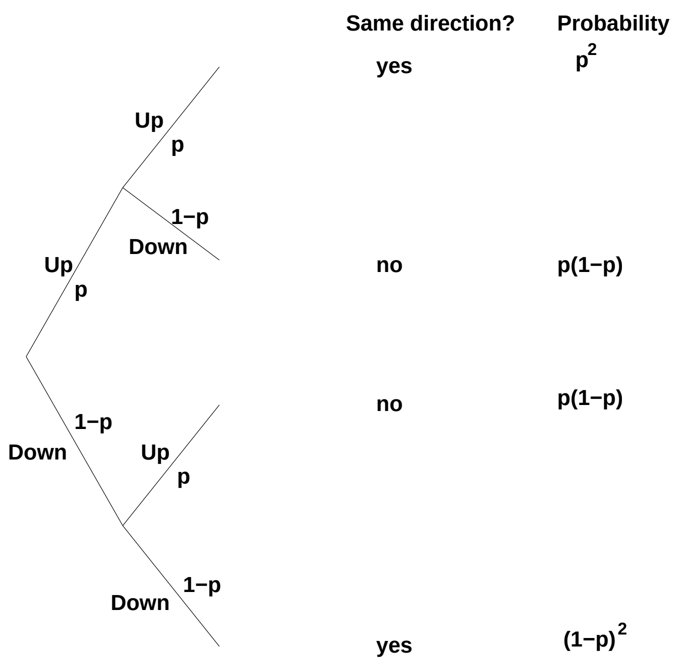
Solution. We can reason as follows:

$$\begin{aligned}
\Pr\{H \mid 6\} &= \frac{\Pr\{H \cap 6\}}{\Pr\{6\}} \\
&= \frac{\Pr\{6 \mid H\} \cdot \Pr\{H\}}{\Pr\{6\}} \\
&= \frac{1/6 \cdot 1/2}{11/72} \\
&= \frac{6}{11}
\end{aligned}$$

Knowing that the result is 6, the probability that heads was flipped is about 54.5%. Since 6 is slightly more likely in the head case than in the tail case, this result is as we would expect.

Problem 8. In Ultimate Frisbee, first possession is determined by an unusual procedure. An opposing player flips two frisbees in the air and then you call either "same" or "different". The frisbees can either fall the same way (both face up or both face down) or different ways (one up and one down). If your call is correct, you win the toss. Assume that the frisbees are identical, but far from fair. Which call should you make and why?

Solution. Let p be the probability that a frisbee falls face up, and S be the event "The frisbees fall the same way". We can draw a tree diagram that shows the different outcomes.

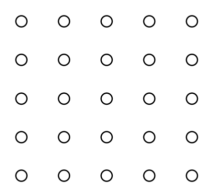


This shows:

$$\begin{aligned}\Pr\{S\} &= p^2 + (1-p)^2 \\ &= 2p^2 - 2p + 1 \\ &= 2\left(p - \frac{1}{2}\right)^2 + \frac{1}{2}\end{aligned}$$

From the last expression, we can conclude that $\Pr\{S\} > \frac{1}{2}$ whenever $p \neq \frac{1}{2}$. So it would be a good idea to call "same", if you want first possession.

Problem 9. There is a 100×100 grid of points, a larger version of the grid shown below:



Suppose that you pick n points from this grid independently and uniformly at random. (You might pick the same point more than once.) Two points are said to be *conflicting* if they lie in either the same row or the same column. What is the smallest value of n such that the probability that there are two conflicting points is greater than $\frac{1}{2}$?

Solution. This is similar to the birthday paradox problem. We could solve it using similar reasoning, however we can actually reuse our earlier result. The key idea is that picking a point uniformly at random in this grid is equivalent to picking a row and a column independently and uniformly at random.

We can restate the problem from this new perspective. Pick n rows and n columns, independently and uniformly at random. If some row is picked more than once or some column is picked more than once, then we have a conflict. Otherwise, there is no conflict. What is the smallest value of n such that the probability of a conflict is greater than $\frac{1}{2}$?

We now have two independent birthday problems, one on the columns and one on the rows. Let C be the event that no column is picked more than once, let R be the event that no row is picked more than once. The probablity that there is no conflict is:

$$\begin{aligned}\Pr\{\text{no conflict}\} &= \Pr\{C \cap R\} \\ &= \Pr\{C\} \cdot \Pr\{R\} \\ &= (\Pr\{C\})^2 \\ &= \left(\prod_{i=1}^{n-1}\left(1 - \frac{i}{100}\right)\right)^2\end{aligned}$$

The second equation uses the fact that rows and columns are selected independently, and so C is independent of R. The third equation uses the fact that a conflict between rows is just as likely as a conflict between columns. In the final step, we plug in our solution to the birthday problem.

For $n = 8$, this sum evaluates to about 56.3%. For $n = 9$, it evaluates to about 47.6%. Therefore, the probability that two points in the original game conflict excedes 50% as of $n = 9$.

(Some students assumed that a point in the grid could be picked at most once. However, since the problem states that points are picked *independently*, it must be possible to pick the same point multiple times.)