Problem Set 10

Due: To your tutor on Friday, May 9.

Problem 1. Suppose that we roll n six-sided dice that are fair and independent. Compute the expected value of the highest roll. You might find the following fact from lecture helpful: for a random variable X that takes on values in $\mathbb{N}$:

$$\text{Ex}[X] = \sum_{k=0}^{\infty} \Pr\{X > k\}$$

Solution. Let $X_1, \ldots, X_n$ be the individual die rolls, and define:

$$X = \max(X_1, \ldots, X_n)$$

Our goal is to evaluate:

$$\begin{aligned}
\text{Ex}[X] &= \sum_{k=0}^{\infty} \Pr\{X > k\} \\
&= \sum_{k=0}^{5} \Pr\{X > k\}
\end{aligned}$$

The infinite sum is equal to the finite sum, because the highest roll can not possibly be greater than 6; thus, all later terms are zero. All that remains is to compute $\Pr\{X > k\}$, which we can do as follows:

$$\begin{aligned}
\Pr\{X > k\} &= 1 - \Pr\{X \leq k\} \\
&= 1 - \Pr\{X_1 \leq k \cap \ldots \cap X_n \leq k\} \\
&= 1 - \Pr\{X_1 \leq k\} \cdot \ldots \cdot \Pr\{X_n \leq k\} \\
&= 1 - \Pr\{X_1 \leq k\}^n \\
&= 1 - \left(\frac{k}{6}\right)^n
\end{aligned}$$

In the first step, we switch to analyzing the complementary event. Next, we observe the highest roll is less than or equal to k if and only if every individual roll is less than or equal to k. The third step uses independence, and the final step uses the fact that each roll has an equal probability of being less than or equal to k. The final step is valid for integral k between 0 and 6.

Substituting this result into the expectation formula gives:

$$\begin{aligned} \operatorname{Ex}[X] &= \sum_{k=0}^{5} \Pr\{X > k\} \\ &= \sum_{k=0}^{5} \left(1 - \left(\frac{k}{6}\right)^n\right) \\ &= 6 - \frac{1^n + 2^n + 3^n + 4^n + 5^n}{6^n} \end{aligned}$$

Problem 2. The starship *Almost Invincible* is cruising across the galaxy. Each day, there is a 1 in 80 chance that the ship is blown up by Arcturian raiders, a 1 in 300 chance that it is eaten by a giant spaceworm, and 1 in 2000 chance that it is sucked into a black hole. Assume that these events are independent. What is the expected duration of the *Almost Invincible*'s cruise?

Solution. Each day, the ship is destroyed with probability:

$$1 - \left(1 - \frac{1}{80}\right) \cdot \left(1 - \frac{1}{300}\right) \cdot \left(1 - \frac{1}{2000}\right) = \frac{781621}{48000000}$$

Therefore, the expected time until the ship is destroyed is:

$$\frac{48000000}{781621} \approx 61.41 \text{ days}$$

Problem 3. For some MIT students, undergraduate life is a continuing struggle against a creature in the closet known as the *Dirty Clothes Monster*. Each term, the monster either is *under control* or else is *out of control*. When a student first arrives at MIT, the monster is under control. Thereafter, the struggle proceeds as follows:

If the monster is under control in a given term, then:

- With probability $\frac{6}{10}$, the monster is under control the next term.

- With probability $\frac{3}{10}$, the monster is out of control the next term.
- With probability $\frac{1}{10}$, the student wins the struggle by graduating.

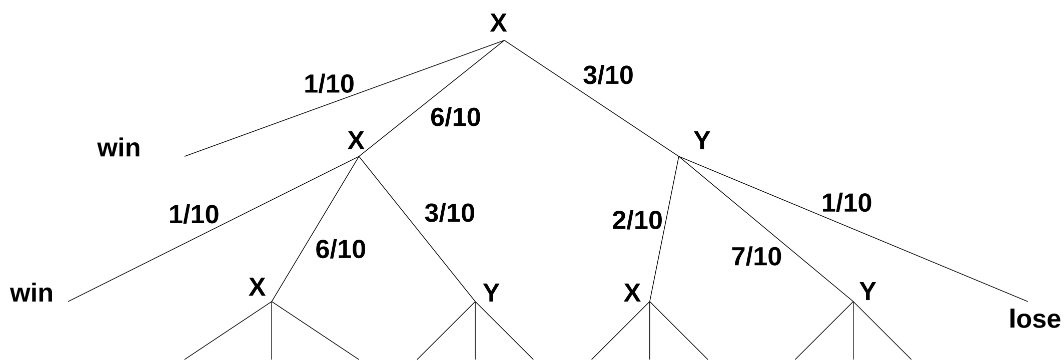
If the monster is out of control in a given term, then:

- With probability $\frac{7}{10}$, the monster is out of control the next term.
- With probability $\frac{2}{10}$, the monster is under control the next term.
- With probability $\frac{1}{10}$, the student is never heard from again.

Let's analyze the prospects of such a student.

(a) Draw a tree diagram deep enough so that you understand the overall structure of the tree. Mark each vertex either X or Y according to whether the monster is under control (X) or out of control (Y).

Solution.



All the X nodes are roots of identical subtrees, and all the Y nodes are roots of identical subtrees.

(b) Let X be the probability that the student eventually wins, given that the monster is currently under control. Let Y be the probability that the student eventually wins, given that the monster is currently out of control. Obtain two equations from your tree diagram, one expressing X in terms of X and Y and one expressing Y in terms of X and Y.

Solution.

$$\begin{aligned} X &= \frac{1}{10} + \frac{6}{10}X + \frac{3}{10}Y \\ Y &= \frac{2}{10}X + \frac{7}{10}Y \end{aligned}$$

(c) Solve the two equations you obtained. What is the probability that a newly-arrived MIT student ultimately triumphs over the Dirty Clothes Monster?

Solution. The solution to the equations is $X = \frac{1}{2}$ and $Y = \frac{1}{3}$. Therefore, the student survives with probability $\frac{1}{2}$.