Problem Set 1

Due: Start of class on February 11.

Problem 1. Suppose that you flip four coins. Assume that each coin is equally likely to come up heads or tails, regardless of how the other three coins turn up. The goal of this problem is to determine which of the following two events is more likely:

1. Two coins come up one way and two come up the other.
2. Three coins come up one way and one comes up the other.

(a) What is the sample space for this experiment?

Solution. Since this experiment has such a regular structure, we'll dispense with the tree diagram.

The sample space consists of sixteen outcomes, each of which specifies the outcomes of the four coin flips:

$$\left\{ \begin{array}{cccc} HHHH & HHHT & HHTH & HHTT \\ HTHH & HTHT & HTTH & HTTT \\ THHH & THHT & THTH & THTT \\ TTHH & TTHT & TTTH & TTTT \end{array} \right\}$$

The outcome $HHHT$, for example, corresponds to the case where the first three coins come up heads and the last comes up tails.

(b) What subsets of this sample space constitute the two events listed above?

Solution. The first event consists of outcomes containing an equal number of T's and H's. There are six of these:

$$\{HHTT, HTHT, HTTH, THHT, THTH, TTHH\}$$

The second event consists of outcomes containing three of one orientation and one of the other. There are eight of these:

$$\{HTTT, THTT, TTHT, TTTH, THHH, HTHH, HHTH, HHHT\}$$

(c) What is the probability of each outcome in the sample space?

Solution. Each outcome has probability 1/16.

(d) What are the probabilities of the two events?

Solution. Recall that the probability of an event is defined to be the sum of the probabilities of the outcomes in that event. Therefore, we have:

$$\begin{aligned}
&\Pr(\text{two one way, two the other}) \\
&\quad = \Pr\{HHTT\} + \Pr\{HTHT\} + \Pr\{HTTH\} \\
&\qquad + \Pr\{THHT\} + \Pr\{THTH\} + \Pr\{TTHH\} \\
&\quad = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} \\
&\quad = \frac{3}{8}
\end{aligned}$$

$$\begin{aligned}
&\Pr(\text{three one way, one the other}) \\
&\quad = \Pr\{HTTT\} + \Pr\{THTT\} + \Pr\{TTHT\} + \\
&\qquad \Pr\{TTTH\} + \Pr\{THHH\} + \Pr\{HTHH\} + \\
&\qquad \Pr\{HHTH\} + \Pr\{HHHT\} \\
&\quad = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} \\
&\quad = \frac{1}{2}
\end{aligned}$$

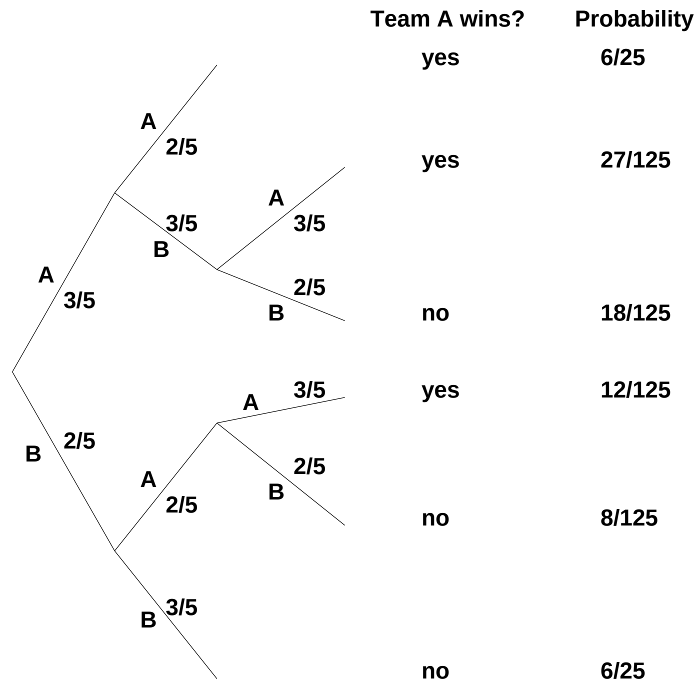
(e) Which event is more likely?

Solution. The second event, that three coins come up one way and one comes up the other, is more likely.

Problem 2. A tournament consists of a sequence of matches between Team A and Team B. The winner of the tournament is the first team to win a total of two matches. In each match, the home team wins with probability 0.6 and the away team wins with probability 0.4, regardless of the outcomes of other matches. The first match is played on Team A's home field, the second match on Team B's field, and the third match (if there is one) on Team A's field again. We want to determine the probability that Team A wins the tournament.

(a) What is the sample space for this experiment?

Solution. A tree diagram for this experiment is shown below.



The sample space consists of six outcomes:

$$\{AA, ABA, ABB, BAA, BAB, BB\}$$

The outcome BAB, for example, corresponds to the situation where Team B wins the first game, Team A wins the second, and Team B wins the third.

(b) What subset of the sample space constitutes the event that Team A wins the tournament?

Solution. The event that Team A wins the tournament consists of three outcomes:

$$\{AA, ABA, BAA\}$$

(c) What is the probability of each outcome in the sample space?

Solution. The outcome probabilities are indicated in the tree diagram above.

(d) What is the probability of the event that Team A wins the tournament?

Solution.

Since the probability of an event is the sum of the outcome probabilities, we have:

$$\begin{aligned}\Pr\{\text{Team A wins}\} &= \Pr\{AA\} + \Pr\{ABA\} + \Pr\{BAA\} \\ &= \frac{6}{25} + \frac{27}{125} + \frac{12}{125}\end{aligned}$$

Problem 3. For each of the following relations, determine whether it is reflexive, whether it is symmetric, and whether it is transitive.

(a) The relation "x is an ancestor of y" on the set of humans.

Solution. Reflexive: no Symmetric: no Transitive: yes

(b) The relation "x is taking a class that y is taking" on the set of MIT students.

Solution. Reflexive: yes Symmetric: yes Transitive: no

(The relation is not transitive because students A and B might be taking one class together, B and C might be taking another class together, but A and C are taking no classes together.)

(c) The relation "x is an integral multiple of y" on the set of positive integers.

Solution. Reflexive: yes Symmetric: no Transitive: yes

(d) The relation "x was born in the same month as y" on the set of 6.042 students.

Solution. Reflexive: yes Symmetric: yes Transitive: yes

(Since all three properties hold, this is an *equivalence relation.*)

(e) The relation defined below on the set $\{1, 2, 3\}$.

$$\{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1)\}$$

Solution. Reflexive: yes Symmetric: yes Transitive: no

(The relation is not transitive because $(3, 1)$ and $(1, 2)$ are in the relation, but $(3, 2)$ is not.)

Problem 4. Consider the following false claim and invalid proof:

False Claim *If a relation R on a set S is symmetric and transitive, then it is also reflexive.*

Invalid Proof. Let x be an arbitrary element of S. Choose another element $y \in S$ such that xRy holds. By symmetry, yRx holds as well. By transitivity, it follows that xRx. $\square$

(a) Describe a relation that is symmetric and transitive, but not reflexive. Your relation may be a familiar one (as in parts (a) - (d) in the preceding problem) or a new one that you define from scratch (as in part(e)).

Solution.

The relation "x was born in the same month as y" on the set of 6.042 students together with the color purple. The relation is not reflexive, because the color purple was not born in the same month as itself; it was not born at all! On the other hand, the relation remains both symmetric and transitive.

(b) Pinpoint the error in the proof as precisely as you can.

Solution. The error arises in the statement, "Choose another element $y \in S$ such that xRy holds." It may be the case that there exists an x for which no such element y exists.

This is precisely the loophole we exploited in the counterexample above. We adjoined a new element (the color purple) to the set of 6.042 students. The new element is unrelated to itself or any other element already in the set. Thus, we have an x such that xRy never holds.

Problem 5. Let A, B, and C be sets. Assume that $A \cap B \cap C = \emptyset$. Prove that:

$$A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A)$$

(If X and Y are sets, then $X - Y$ denotes the set of elements in X that are not also in Y.)

Solution. We prove the equality by showing that the left set contains the right set and vice versa. Let S be the set on the left, $A \cup B \cup C$, and let T be the set on the right, $(A - B) \cup (B - C) \cup (C - A)$. In these terms, our objective is to prove that $S = T$.

First, we show that S is contained in T. Let x be an element of $S = A \cup B \cup C$. There are then six cases:

- $x \in A, x \notin B, x \notin C$: Since $x \in A$ and $x \notin B$, we have $x \in A - B$, and so $x \in T$.
- $x \notin A, x \in B, x \notin C$: Since $x \in B$ and $x \notin C$, we have $x \in B - C$, and so $x \in T$.
- $x \notin A, x \notin B, x \in C$: Since $x \in C$ and $x \notin A$, we have $x \in C - A$, and so $x \in T$.
- $x \in A, x \in B, x \notin C$: Since $x \in B$ and $x \notin C$, we have $x \in B - C$, and so $x \in T$.
- $x \in A, x \notin B, x \in C$: Since $x \in A$ and $x \notin B$, we have $x \in A - B$, and so $x \in T$.
- $x \notin A, x \in B, x \in C$: Since $x \in C$ and $x \notin A$, we have $x \in C - A$, and so $x \in T$.

In every case, $x \in T$ as well; therefore, every element of S is also an element of T, and so $S \subseteq T$. (Note that the case $x \in A, x \in B, x \in C$ does not exist, since $A \cap B \cap C = \emptyset$ by assumption.)

Now we show that T is contained in S. Let x be an element of $T = (A-B) \cup (B-C) \cup (C-A)$. There are then three cases:

- $x \in A - B$: This implies that $x \in A$, and so $x \in S$.
- $x \in B - C$: This implies that $x \in B$, and so $x \in S$.
- $x \in C - A$: This implies that $x \in C$, and so $x \in S$.

In every case, $x \in S$ as well; therefore, every element of T is also an element of S, and so $T \subseteq S$. Since set S contains set T and vice versa, S and T must be the same set; that is, $S = T$.

Problem 6. The following problems require you to make back-of-the-envelope calculations. While these problems are contrived, the ability to make estimates and rough calculations can serve you well in checking your work and in thinking critically about the world at large. Later, the mathematical tools that we develop will increase the number of situations that you can analyze. For these problems, try to give reasonably accurate answers. Show your assumptions and the calculations that lead to your conclusions. You will probably have to search for some information online; *please cite your sources.*

(a) Could the entire world population, working together, drink Niagara Falls? In other words, which is greater: the average amount of water that flows over the falls each day or the average amount of water that the world population drinks each day?

Solution.

The average flow rate of Niagara Falls is 212,000 cubic feet of water per second.[1] The average amount of water that flows over the falls each day is therefore:

$$212,000 \frac{\text{ft}^3}{\text{sec}} \cdot 60 \frac{\text{sec}}{\text{min}} \cdot 60 \frac{\text{min}}{\text{hr}} \cdot 24 \frac{\text{hr}}{\text{day}} \quad = \quad 18,316,800,000 \frac{\text{ft}^3}{\text{day}}$$

Using 6.25 billion for the world population[2], we can compute the necessary consumption rate per person:

$$\frac{18,316,800,000 \frac{\text{ft}^3}{\text{day}}}{6.25 \cdot 10^9 \text{ people}} \approx 3 \frac{\text{ft}^3}{\text{person} \cdot \text{day}}$$

Drinking 3 cubic feet of water per day is far beyond the human norm, so the answer is *no*.

(b) How long would you have to type to fill a current-generation hard drive? (Assume that only the characters you enter manually are stored; you may not automatically generate files or download *The Two Towers* from Bearshare.)

Solution.

A typical hard drive holds 40 gigabytes. A fast typist can produce about 100 words per minute or, say, 500 bytes per minute. This implies a typing time of about:

[1] http://www.iaw.com/~falls/origins.html#Power

[2] http://blue.census.gov/cgi-bin/ipc/popclockw

$$\frac{40 \text{ gigabytes}}{500 \frac{\text{bytes}}{\text{min}}} = 8 \cdot 10^7 \text{ min}$$

Converting to years, we find:

$$\frac{8 \cdot 10^7 \text{ min}}{60 \frac{\text{min}}{\text{hr}} \cdot 24 \frac{\text{hr}}{\text{day}} \cdot 365 \frac{\text{day}}{\text{yr}}} = 152 \text{ years}$$

(c) Is the United States more or less densely populated than the world as a whole, excluding Antarctica?

Solution. The land area of the United States is about 9.2 million square kilometers, and the population of the United States is around 280 million[3]. Therefore, the population density of the United States is around:

$$\frac{280 \cdot 10^6 \text{ people}}{9.2 \cdot 10^6 \text{ km}^2} = 30 \frac{\text{people}}{\text{km}^2}$$

On the other hand, the total amount of land on earth is about 150 million square kilometers[4], of which about 13 million square kilometers is Antarctica[5]. The population of the world is about 6.25 billion, so we have a world population density, excluding Antarctica, of about:

$$\frac{6.25 \cdot 10^9 \text{ people}}{150 \cdot 10^6 \text{ km}^2 - 13 \cdot 10^6 \text{ km}^2} = 46 \frac{\text{people}}{\text{km}^2}$$

Therefore, the United States is much more sparsely populated than the world as a whole.

(d) Let's face it: Massachusetts is wretched in winter, but Hawaii is wonderful. How long would it take to transport Mauna Loa to central Massachusetts, if all the dirt and rock entering the state must be brought in by dump truck along the Mass Pike? (Assume that the Hawaiians will helpfully bring material to our state line as fast as we can pick it up.)

Solution. The volume of Mauna Loa is about 70,000 cubic kilometers or $7 \cdot 10^{13}$ cubic meters[6]. A really huge dump truck can carry about 200 cubic meters of material[7]. Therefore, the number of truckloads needed would be:

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[3] http://www.cia.gov/cia/publications/factbook/geos/us.html

[4] http://www.cia.gov/cia/publications/factbook/geos/xx.html#Geo

[5] http://www.scar.org/Antarctic%20Info/Ant%20stats.html

[6] http://gsa.confex.com/gsa/2002CD/finalprogram/abstract_34712.htm

[7] http://www.equipmentcentral.com/north_america/new_equipment/machine_list.cfm?machine_type_id=&ecweb=KA_Trucks

$$\frac{7 \cdot 10^{13} \frac{\text{m}^3}{\text{mountain}}}{200 \frac{\text{m}^3}{\text{truck}}} = 3.5 \cdot 10^{11} \frac{\text{trucks}}{\text{mountain}}$$

Let's optimistically suppose we roll these monsters across the border at a rate of one per second. The number of seconds per year is:

$$60 \frac{\text{sec}}{\text{min}} \cdot 60 \frac{\text{min}}{\text{hr}} \cdot 24 \frac{\text{hr}}{\text{day}} \cdot 365 \frac{\text{day}}{\text{year}} = 3.2 \cdot 10^7$$

Dividing these quantities suggests that we'll need about 10,000 years to get the job done.

(e) It is often said that the Great Wall of China is the only man-made structure visible from the moon. From how far away would you have to view a human hair so that its apparent width to you would be the same as the apparent width of the Great Wall to someone on the moon? Is the original claim reasonable?

Solution. The Great Wall is about 10 meters across, and 350,000 km from the moon[8]. A human hair is about 100 microns across or about 100,000 times narrower than the Wall[9]. Thus, the Great Wall is about as visible as a human hair 3.5 km away. The claim is false.

(f) It is often said that human activity may significantly raise the concentration of carbon dioxide in the atmosphere, leading to global warming. But the planet is huge; only a tiny percentage of the atmosphere is carbon dioxide, but that still amounts to *750 gigatons*. Suppose, pessimistically, that everyone on earth drives like a typical American. What would be the total mass of carbon dioxide released each year by automobiles? Is the original claim reasonable?

Solution. Burning a gallon of gasoline burns about 20 pounds of carbon dioxide[10]. If a typical driver covers about 10,000 miles per year at 20 miles per gallon, then he or she consumes about 500 gallons of gasoline per year. Suppose that 4 billion people drive this much each year. Then the annual carbon dioxide output from cars alone is:

$$\begin{aligned} 20 \frac{\text{lbs CO2}}{\text{gal}} \cdot 500 \frac{\text{gal}}{\text{driver}} \cdot 4 \cdot 10^9 \text{ drivers} &= 4 \cdot 10^{13} \text{ lbs CO2} \\ &= 20 \text{ gigatons CO2} \end{aligned}$$

[8] http://stardate.org/resources/ssguide/moon.html
[9] http://hypertextbook.com/facts/1999/BrianLey.shtml
[10] http://globalwarming.enviroweb.org/ishappening/sources/sources_co2_facts3.html

Such human activity could significantly raise the concentration of carbon dioxide in the atmosphere, albeit over a period of decades. This is consistent with empirical observations done at the Mauna Loa Observatory, whose days are numbered approximately above.