# Lecture 3 - Monty Hall and Ill-Posed Problems

6.042 - February 11, 2003

## 1 The Monty Hall Problem

In the 1970's, there was a game show called *Let's Make a Deal*, hosted by Monty Hall and his assistant Carol Merrill. At one stage of the game, a contestant is shown three doors. The contestant knows there is a prize behind one door and that there are goats behind the other two. The contestant picks a door. To build suspense, Carol always opens a *different* door, revealing a goat. The contestant can then stick with his original door or switch to the other unopened door. He wins the prize only if he now picks the correct door. Should the contestant "stick" with his original door, "switch" to the other door, or does it not matter?

This was the subject of an "Ask Marilyn" column in *Parade* Magazine a few years ago. Marilyn wrote that your chances of winning were 2/3 if you switched— because if you switch, then you win if the prize was originally behind either of the two doors you didn't pick. Now, Marilyn has been listed in the *Guiness Book of World Records* as having the world's highest IQ, but for this answer she got a tidal wave of critical mail, some of it from people with Ph.D.'s in mathematics, telling her she was wrong. Most of her critics insisted that the answer was 1/2, on the grounds that the prize was equally likely to be behind each of the remaining, closed doors. The pros and cons of these arguments still stimulate debate.

It turned out that Marilyn was right[1] But given the debate, it is clearly not apparent which of the intuitive arguments for 2/3 or 1/2 is reliable. Rather than try to come up with our own explanation in words, let's use our standard approach to finding probabilities. In particular, we will analyze the probability that the contestant wins with the "switch" strategy; that is, the contestant chooses a random door initially and then always switches after Carol reveals a goat behind one door. We break the problem down into the standard four steps.

### Step 1: Find the Sample Space

In the Monty Hall problem, an outcome is a triple of door numbers:

1. The number of the door concealing the prize.



[1]Rather, she gave the right answer to the question posed here, which is the one she intended to ask. Unfortunately for her, she actually presented an *ill-posed* question for which there is no meaningful answer. See the next section for examples of such questions.

2. The number of the door initially chosen by the contestant.

3. The number of the door Carol opens to reveal a goat.

For example, the outcome $(2, 1, 3)$ represents the case where the prize is behind door 2, the contestant initially chooses door 1, and Carol reveals the goat behind door 3. In this case, a contestant using the "switch" strategy wins the prize.

Not every triple of numbers is an outcome; for example, $(1, 2, 1)$ is not an outcome, because Carol never opens the door with the prize. Similarly, $(1, 2, 2)$ is not an outcome, because Carol does not open the door initially selected by the contestant, either.

The tree diagram for the Monty Hall problem is shown in Figure 1. As usual, each vertex in the tree corresponds to a state of the experiment. In particular, the root represents the initial state, before the prize is even placed. Internal nodes represent intermediate states of the experiment, such as after the prize is placed, but before the contestant picks a door. Each leaf represents a final state, an outcome of the experiment. One can think of the experiment as a walk from the root (initial state) to a leaf (outcome). In the figure, each leaf of the tree is labeled with an outcome (a triple of numbers) and a "W" or "L" to indicate whether the contestant wins or loses.

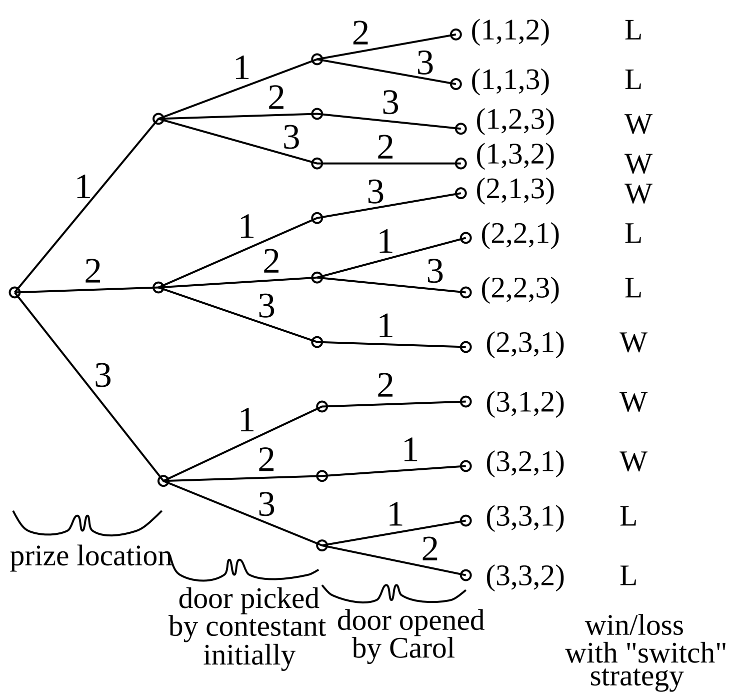



Figure 1: This is a tree diagram for the Monty Hall problem. Each of the 12 leaves of the tree represents an outcome. A "W" next to an outcome indicates that the contestant wins, and an "L" indicates that he loses.

## Step 2: Define Events of Interest

For the Monty Hall problem, let $S$ denote the sample space, the set of all 12 outcomes shown in Figure 1. The event $W \subseteq S$ that the contestant wins with the "switch" strategy consists of six outcomes:

$$W = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$$

The event $L \subset S$ that the contestant loses is the complementary set:

$$L = \{(1,1,2),(1,1,3),(2,2,1),(2,2,3),(3,3,1),(3,3,2)\}$$

Our goal is to determine the probability of the event $W$; that is, the probability that the contestant wins with the "switch" strategy.

Well, the contestant wins in 6 outcomes and loses in 6 outcomes. Does this not imply that the contestant has a $6/12 = 1/2$ chance of winning? No! Under reasonable assumptions, this sample space is not uniform! Some outcomes are more likely than others. We must compute the probability of each outcome.

## Step 3: Compute Outcome Probabilities

### 1.1 Assumptions

To assign a meaningful probability to each outcome in the Monty Hall problem, we must make some assumptions. The following three are sufficient:

1. The prize is placed behind each door with probability 1/3.
2. No matter where the prize is placed, the contestant picks each door with probability 1/3.
3. No matter where the prize is placed, if Carol has a choice of which door to open, then she opens each possible door with equal probability.

The first two assumptions capture the idea that the contestant initially has no idea where the prize is placed. The third assumption eliminates the possibility that Carol somehow secretly communicates the location of the prize by which door she opens. Assumptions of this sort almost always arise in probability problems; making them explicit is a good idea, although in fact not all of these assumptions are absolutely necessary. For example, it doesn't matter how Carol chooses a door to open in the cases when she has a choice, though we won't prove this.

## 1.2 Assigning Probabilities to Outcomes

With these assumptions, we can assign probabilities to outcomes in the Monty Hall problem by a calculation illustrated in Figure 2 and described below. There are two steps.

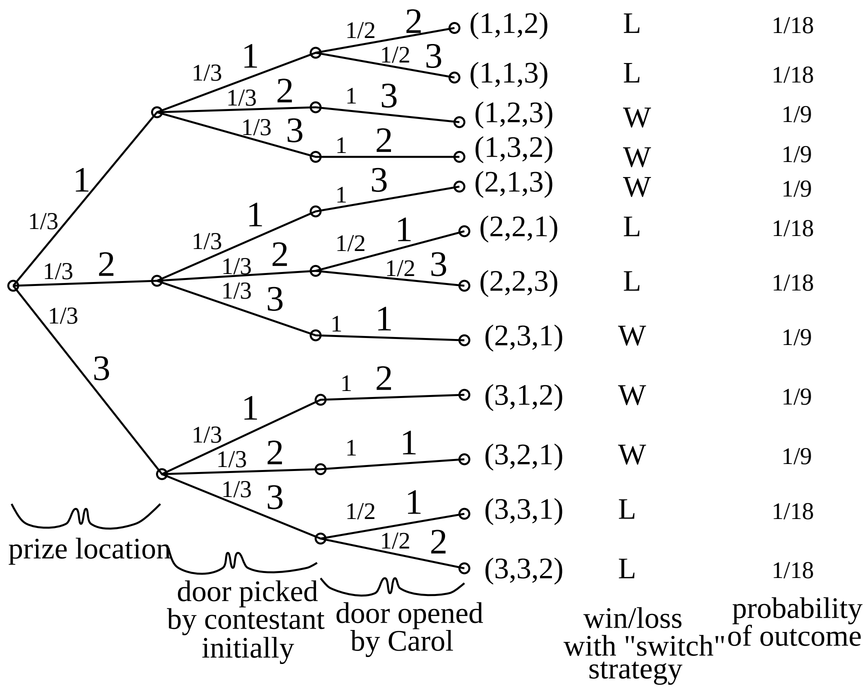



Figure 2: This is the tree diagram for the Monty Hall problem, annotated with probabilities for each outcome.

The first step is to record a probability on each edge in the tree diagram. Recall that each node represents a state of the experiment, and the whole experiment can be regarded as a walk from the root (initial state) to a leaf (outcome). The probability recorded on an edge is the probability of moving from the state corresponding to the parent node to the state corresponding to the child node. These edge probabilities follow from our three assumptions about the Monty Hall problem.

Specifically, the first assumption says that there is a 1/3 chance that the prize is placed behind each of the three doors. This gives the 1/3 probabilities on the three edges from the root. The second assumption says that no matter how the prize is placed, the contestant opens each door with probability 1/3. This gives the 1/3 probabilities on edges leaving the second layer of nodes. Finally, the third assumption is that if Carol has a choice of what door to open, then she opens each with equal probability. In cases where Carol has no choice, edges from the third layer of nodes are labeled with probability 1. In cases where Carol has two choices, edges are labeled with probability 1/2.

The second step is to use the edge weights to compute a probability for each outcome by multiplying the probabilities along the edges leading to the outcome. This way of assigning probabilities reflects our idea that probability measures the fraction of times that a given outcome should happen over the course of many experiments. Suppose we want the probability of outcome $(2, 1, 3)$. In $1/3$ of the experiments, the prize is behind the second door. Then, in $1/3$ of these experiments when the prize is behind the second door, and the contestant opens the first door. After that, Carol has no choice but to open the third door. Therefore, the probability of the outcome is the product of the edge probabilities, which is

$$\frac{1}{3} \cdot \frac{1}{3} \cdot 1 = \frac{1}{9}.$$

For example, the probability of outcome $(2, 2, 3)$ is the product of the edge probabilities on the path from the root to the leaf labeled $(2, 2, 3)$. Therefore, the probability of the outcome is

$$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{18}.$$

Similarly, the probability of outcome $(3, 1, 2)$ is

$$\frac{1}{3} \cdot \frac{1}{3} \cdot 1 = \frac{1}{9}.$$

The other outcome probabilities are worked out in Figure 2.

## Step 4: Compute Event Probabilities

We now have a probability for each outcome. All that remains is to compute the probability of $W$, the event that the contestant wins with the "switch" strategy. The probabilility of an event is simply the sum of the probabilities of all the outcomes in it. So the probability of the contestant winning with the "switch" strategy is the sum of the probabilities of the six outcomes in event $W$, namely, $2/3$:

$$\begin{aligned}
\Pr\{W\} &= \Pr\{(1,2,3)\} + \Pr\{(1,3,2)\} + \Pr\{(2,1,3)\} + \\
&\quad \Pr\{(2,3,1)\} + \Pr\{(3,1,2)\} + \Pr\{(3,2,1)\} \\
&= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} \\
&= \frac{2}{3}
\end{aligned}$$

In the same way, we can compute the probability that a contestant loses with the "switch" strategy. This is the probability of event $L$:

$$
\begin{aligned}
\Pr\{L\} &= \Pr\{(1,1,2)\} + \Pr\{(1,1,3)\} + \Pr\{(2,2,1)\} + \\
&\quad \Pr\{(2,2,3)\} + \Pr\{(3,3,1)\} + \Pr\{(3,3,2)\} \\
&= \frac{1}{18} + \frac{1}{18} + \frac{1}{18} + \frac{1}{18} + \frac{1}{18} + \frac{1}{18} \\
&= \frac{1}{3}
\end{aligned}
$$

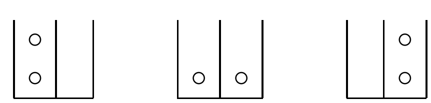
The probability of the contestant losing with the switch strategy is $1/3$. This makes sense; the probability of winning and the probability of losing ought to sum to 1!

We can determine the probability of winning with the "stick" strategy without further calculations. In every case where the "switch" strategy wins, the "stick" strategy loses, and vice versa. Therefore, the probability of winning with the stick strategy is $1 - 2/3 = 1/3$.

Solving the Monty Hall problem formally requires only simple addition and multiplication. But trying to solve the problem with "common sense" leaves us running in circles!

# 2 Ill-Posed Problems

Suppose that I throw two balls into three bins at random. What is the probability that both balls end up in the same bin? As shown below, three configurations can result:



Often when people say "at random", they implicitly mean "uniformly at random", so perhaps we should assume that these configurations are equally likely. That suggests that the answer is $2/3$.

On the other hand, perhaps we should assume that *each* ball is thrown into a bin uniformly at random, without regard to where the other ball ends up. Under this model, we obtain the tree diagram shown in Figure 3. The event that both balls fall into the same bin consists of the outcomes $(1,1)$ and $(2,2)$, both of which have probability $\frac{1}{4}$. This suggests that the probability that both balls fall into the same bin is actually $\frac{1}{2}$.

Which answer is right? Both are defensible. Given the statement of the problem, there are at least two reasonable interpretations that give rise to two different probabilistic models and give two contradictory answers. The question itself is *ill-posed.*

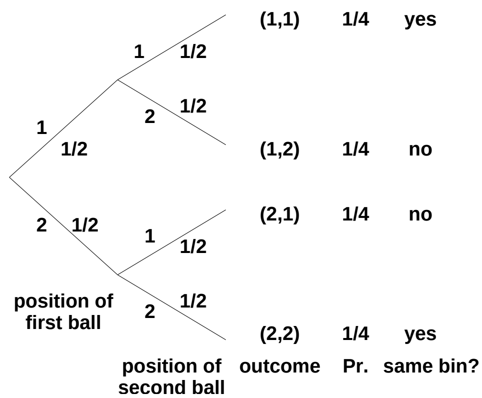
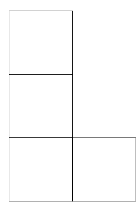



Figure 3: The tree diagram for the experiment of throwing two balls into two bins, where each ball is thrown uniformly at random and without regard to the other ball.

## Where are the Chickens?

We have four boxes arranged in an L-shape as shown below. Each box is large enough to hold one chicken. Call the three boxes stacked vertically "the column", and call the two boxes side-by-side "the row".



Suppose that we place two chickens at random, one into the row and one into the column. What is the probability that a chicken ends up in the lower-left box, at the elbow of the L?

The row chicken is in the elbow with probability $\frac{1}{2}$. However, if the row chicken is not in the elbow (also probability $\frac{1}{2}$), then the column chicken is inthe elbow with probability $\frac{1}{3}$. Therefore, there is a chicken in the elbow with probability:

$$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} \quad = \quad \frac{2}{3}$$

On the other hand, we can place the first chicken in any of the four boxes. With probability $\frac{1}{4}$ it is in the elbow and we don't care about the location of the other chicken, because we are assured of having one in the row and one in the column. Otherwise, with probability $\frac{2}{4}$, the first chicken is in the upper part of the $L$. In this case, the second chicken must go in one of the two row boxes, and so it ends up in the elbow with probability $\frac{1}{2}$. Finally, with probability $\frac{1}{4}$, the first chicken is in the right box in the row. In this case, the second chicken must go in one of the three column boxes, and so it falls into the elbow with probability $\frac{1}{3}$. Overall, there is a chicken in the elbow with probability:

$$\frac{1}{4} + \frac{2}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{3} \quad = \quad \frac{7}{12}$$

Both interpretations of the question are reasonable, but the answers are different! Once again, the problem is that the question is ill-posed. There is not enough information about the experiment to determine a unique model and thus a unique answer.

# 3 A Mother with Two Children

Consider the following two questions:

1. A mother has two children. At least one is a boy. What is the probability that the other one is a boy?

2. A mother has two children. The older one is a boy. What is the probability that the other one is a boy?

Letters about such questions were second only those concerning the Monty Hall problem in Marilyn vos Savant's mailbag. The surprising fact is that the answers to the two questions are significantly different. A rigorous explanation will have to wait until we introduce *conditional probability* in the next lecture. However, we can give an intuitive explanation for the difference now.

If a mother has two children, then there are four equally probable outcomes: two boys ($BB$), a boy and then a girl ($BG$), a girl and then a boy ($GB$) or two girls ($GG$).

- In the first experiment, we rule out only one outcome ($GG$), leaving three equally-likely possibilities. In only one of these ($BB$) is the other child a boy. Therefore, the answer is 1/3.

- In the second experiment, we rule out *two* outcomes ($GB$ and $GG$), leaving only two equally-likely possibilities. In one of these ($BB$), the other child is a boy. Therefore, the answer is 1/2.

In this case, the problem is not that either question is ill-posed, but rather that there are two very similar questions that— amazingly— have different answers! In general, specific side information ("the older one is a boy") leads to different conclusions than general side information ("at least one is a boy"). Be aware of this distinction, because it arises in many problems!