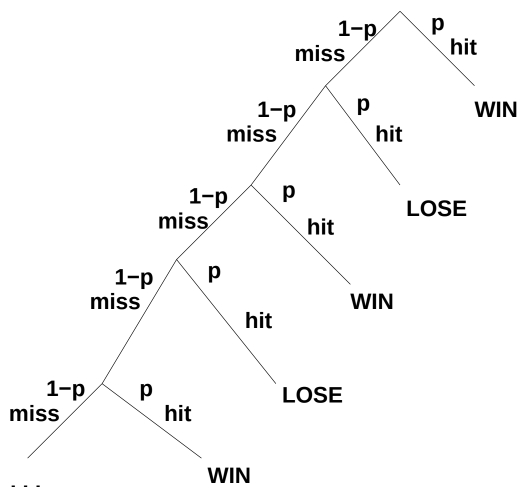
Lecture 21 - Random Processes

6.042 - May 6, 2003

1 The Duel

Two gunfighters take turns shooting at each other. Each shot hits with probability $p = \frac{1}{6}$ and misses with probability $1 - p = \frac{5}{6}$. What is the probability that the first gunfighter to shoot wins the duel?

We can approach this problem by the usual method, drawing a tree diagram. In contrast to previous problems, this tree diagram is infinite. As a practical matter, we can treat an infinite tree in the same way that we treat large finite trees: draw just enough to make the overall structure apparent.



The probability that the first shooter wins is the sum of the probabilities of the outcomes in that event:

$$\begin{aligned}
\Pr\{\text{first shooter wins}\} &= p + (1-p)^2 p + (1-p)^4 p + (1-p)^6 p + \ldots \\
&= p \sum_{k=0}^{\infty} (1-p)^{2k} \\
&= p \cdot \frac{1}{1-(1-p)^2} \\
&= \frac{1}{2-p}
\end{aligned}$$

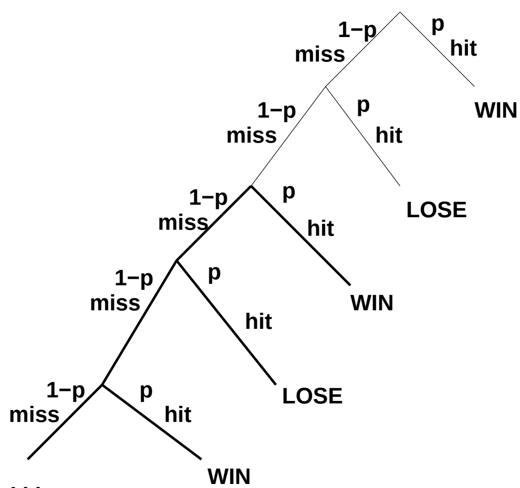
In the second step, we pull out a p from every term. This leaves a geometric sum. The third step uses the formula for an infinite geometric sum with ratio $r = (1-p)^2$:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \qquad \text{(for } |r| < 1\text{)}$$

The last step is simplification. Since each shot hits with probability $p = \frac{1}{6}$, the first gunfighter wins with probability $\frac{6}{11}$.

1.1 A Fixed-Point Approach

There is a versatile, alternative approach to the gunfighter problem. The key idea is to identify subtrees in the tree diagram that strictly contain themselves. With infinite trees, this is possible! For example, notice that the subtree shown below in bold is a complete copy of the tree as a whole.



This reflects the fact that duel is in the same state after two misses that it in was at the very beginning. In each case, both gunfighters are alive and the first gunfighter is about to shoot.

Let's introduce a variable x to denote the probability that the first gunfighter eventually wins, given that the duel is in this state. Then we can read an equation in x directly from the tree:

$$x = p + (1-p)^2 x$$

The p appears because the first gunfighter may hit immediately. Otherwise, with probability $(1-p)^2$, both gunfighters miss and the duel is back in its original state. If this happens, then the first gunfighter eventually wins with probability x. This gives the $(1-p)^2 x$ term. The only remaining possibility is that the first gunfighter misses and the second hits. However, the first gunfighter is dead in that case, so this possbility adds nothing to x, the probability that the first gunfighter wins.

Solving the equation above for x gives:

$$x = \frac{1}{2-p}$$

This is the probability that the first gunfighter wins, given that he is about to shoot. Since this is the situation at the start of the duel, this is also the overall probability that the first gunfighter wins. Happily, this is the same answer that we found before!

2 Infinite Problems

Infinite processes and infinite sample spaces introduce theoretical complications whose resolution lies beyond the scope of this course. We'll note some of those complications and show you some practical methods for working out problems involving infinity, but we're going to gloss over the deeper theoretical issues.

For example, let's work out the sample space for the duel. Let M indicate a missed shot, and let H indicate a hit. Then the sample space is:

$$\{H,\ MH,\ MMH,\ MMMH,\ \ldots\} \cup \{MMM\ldots\}$$

This says that a duel either consists of a sequence of misses followed by a hit or else an infinite sequence of misses. The last outcome is the most puzzling. Each of the finite sequences corresponds to a leaf vertex in the tree diagram, as usual. However, the outcome $MMM\ldots$ corresponds to an *infinite path* through the tree rather than to a leaf. And there is another oddity. In principle, the gunfighters could go on missing forever, much as you could flips heads forever on a fair coin, so $MMM\ldots$ must be included as an outcome in the sample space. But the probability of that outcome is zero!

This leads to a weird conclusion: an event with probability zero *can* occur. For example, suppose that we flip a fair coin forever. The probability of generating any particular sequence of heads and tails (say, HTHTHTHTHTHT...) is zero. And yet we are bound to get some sequence. Therefore, perversely, *we are certain to generate a sequence that has zero probability of being generated.*

Maybe you should revisit those childhood plans to be a firefighter.

2.1 Kolmogorov's Axioms

The Russian mathematician Kolmogorov proposed a retooling of the foundations of probability to cope with such problems. In this course, we regard Pr as a function defined on *outcomes*, individual elements of the sample space. In Kolmogorov's reformulation, Pr is a function defined on *events*, entire subsets of the sample space. In particular, three properties must hold for every valid probability function Pr over a sample space S:

- $\Pr(S) = 1$
- For all events $A \subseteq S$, $\Pr\{A\} \geq 0$.
- If $A_1, A_2, A_3, \ldots$ is a sequence of disjoint events, then:

$$\Pr\{A_1 \cup A_2 \cup A_3 \cup \ldots\} \quad = \quad \Pr\{A_1\} + \Pr\{A_2\} + \Pr\{A_3\} + \ldots$$

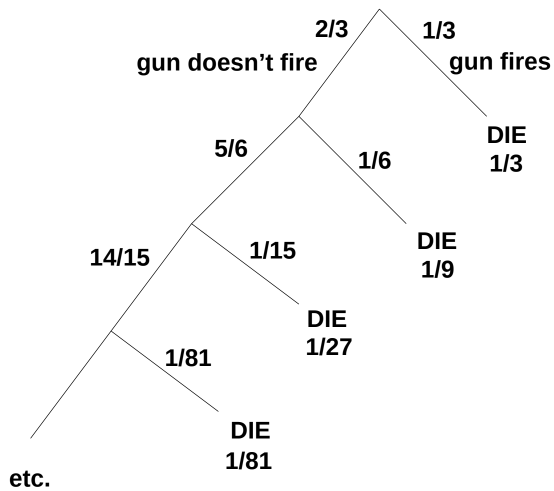
One implication of Kolmogorov's axioms is that *there is no way to pick a natural number uniformly at random*. The proof is by contradiction. Suppose that there is a way to pick a natural number uniformly at random. This means that over some sample space S, we can define a random variable X that takes on each natural number with some fixed probability ϵ. Then we can reason as follows:

$$\begin{aligned}
1 &= \Pr\{S\} \\
&= \sum_{k=0}^{\infty} \Pr\{X = k\} \\
&= \sum_{k=0}^{\infty} \epsilon \\
&= \begin{cases} 0 & \text{if } \epsilon = 0 \\ \infty & \text{if } \epsilon > 0 \end{cases}
\end{aligned}$$

The first step uses Kolmogorov's first property. The second step uses the third property, since the events $X = k$ partition the sample space. The third step uses the fact that the random variable X is equal to each natural number with probability ϵ. In the final step, we evaluate the sum. The result is either 0 (if $\epsilon = 0$) or infinity (if $\epsilon > 0$). By the second property, these are the only two possibilities. Therefore, we have a contradiction. Thus, there does not exist a way to pick a natural number uniformly at random.

2.2 "Prussian" Roulette

Suppose that a lunatic is playing a variant of Russian roulette in which the gun becomes progressively less likely to fire. In particular, let's set up the probabilities so that the lunatic dies after exactly k shots with probability $1/3^k$. Our probability tree might look like this:



The probabilities on the edges are constructed just to make the probabilities on the leaves work out. If the tree continues in this way, then the probability that the lunatic *never* dies is:

$$\begin{aligned} 1-\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots\right) &= 1-\frac{1}{3}\cdot\sum_{k=0}^{\infty}\frac{1}{3^k} \\ &= 1-\frac{1}{3}\cdot\left(\frac{1}{1-1/3}\right) \\ &= \frac{1}{2} \end{aligned}$$

Therefore, there with probability $\frac{1}{2}$, this process *never* terminates!

3 The Duel Revisited

Let's now return to the two gunfighters who take turns shooting at each other, hitting on each shot with probability $p = \frac{1}{6}$. What is the expected number of shots fired during the gunfight?

3.1 An Expectation Formula

To determine the expected duration of the duel, we'll use a nifty new formula for the expectation of a natural-valued random variable.

Theorem 1 *Let X be a random variable that takes on values in $\mathbb{N}$. Then:*

$$Ex[X] = \sum_{k=0}^{\infty} \Pr\{X > k\}$$

Proof. We start with the left side of the equation and reason as follows.

$$\begin{array}{rcccccccc}
\mathrm{Ex}[X] & = & 1 \cdot \Pr\{X = 1\} & + & 2 \cdot \Pr\{X = 2\} & + & 3 \cdot \Pr\{X = 3\} & + & \ldots \\
& = & \Pr\{X = 1\} & + & \Pr\{X = 2\} & + & \Pr\{X = 3\} & + & \ldots \\
& & & + & \Pr\{X = 2\} & + & \Pr\{X = 3\} & + & \ldots \\
& & & & & + & \Pr\{X = 3\} & + & \ldots \\
& & & & & & & + & \ldots \\
& = & \multicolumn{7}{l}{\Pr\{X > 0\} + \Pr\{X > 1\} + \Pr\{X > 2\} + \ldots}
\end{array}$$

The first step uses the definition of expected value. In the second step, we expand each term $k \cdot \Pr\{X = k\}$ to a *column* of terms containing k copies of the term $\Pr\{X = k\}$. In the third step, we collapse each *row* of terms $\Pr\{X = k+1\} + \Pr\{X = k+2\} + \ldots$ to a single term $\Pr\{X > k\}$. The last expression is equal to the right side of the equation, so the theorem is proved. □

3.2 Applying the Formula

Let the random variable X equal the number of shots fired in the duel. Then we have:

$$\begin{aligned}
\mathrm{Ex}[X] &= \sum_{k=0}^{\infty} \Pr\{X > k\} \\
&= \sum_{k=0}^{\infty} (1-p)^k \\
&= \frac{1}{1-(1-p)} \\
&= \frac{1}{p}
\end{aligned}$$

The first step uses the new expectation formula. For the second step, note that more than k shots are fired only if the first k shots are misses, which happens with probability $(1-p)^k$. We use the formula for the sum of a geometric series in the third step, and the final step is simplification. Since our gunfighters hit with probability $p = 1/6$, the expected number of shots fired is $1/p = 6$.

Restating this conclusion in more general terms gives a fact that is very useful and worth remembering. Suppose that there is a sequence of independent trials, each of which succeeds with probability p and fails with probability $1-p$. Then the expected number of trials needed to obtain a success is $1/p$.

4 A Good Bet

A big part of our job as 6.042 staff is to constantly deceive, cheat, and abuse students. But this time we'll make an exception! How much would you pay us up front for the privilege of playing the following game? You flip a coin until you get heads. Then we pay you 2^k dollars, where k is the total number of times you flipped. For example, if you flipped TTH, then we would pay you \$8.

Let the random variable X be the amount of money that we pay you, if you play this game.

$$
\begin{aligned}
\operatorname{Ex}[X] &= \sum_{k=1}^{\infty} 2^k \cdot \Pr\{\text{you flip } k \text{ times}\} \\
&= \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k} \\
&= \sum_{k=1}^{\infty} 1 \\
&= \infty
\end{aligned}
$$

The first step uses the definition of expectation. In the second step, note that you flip exactly k times only if you flip $k-1$ tails followed by a head, which happens with probability $1/2^k$. In the remaining two steps, we simplify and observe that the sum diverges.

This calculation shows that your expected payoff is *infinite*! At least in theory, this is the deal of a lifetime; you should be willing to pay everything you've got for just one chance to play this game!

4.1 Some Practical Considerations

The theoretical analysis of this problem overlooks some important practical considerations.

This game looks good on paper because you have a tiny probability of winning a vast amount of money. Mathematically, this makes the expected payoff infinite. But, in practice, winnings obey a law of diminishing returns. For example, would you rather have a million dollars upfront or a 1-in-a-1000 chance of winning a billion dollars? (And, correspondingly, a 999-in-a-1000 chance of getting nothing.) Most people would probably take the million, figuring that while a billion dollars is a 1000 times *more*, it is unlikely to make them 1000 times *happier*; either way, you're rich! So why not take the 100% chance rather than the 0.1% chance?

Futhermore, there isn't an infinite amount of money in the world and certainly isn't an infinite amount in our pockets. Suppose that you suspect we actually couldn't pay you more than $2^{20} \approx$ a million dollars, no matter how many tails you flip. In that case, we can compute the expected value of the game as follows:

$$\begin{aligned}
\mathrm{Ex}[X] &= \left(\sum_{k=1}^{20} 2^k \cdot \Pr\{\text{you flip } k \text{ times}\}\right) + \left(2^{20} \cdot \Pr\{\text{you flip more than 20 times}\}\right) \\
&= \left(\sum_{k=1}^{20} 2^k \cdot \frac{1}{2^k}\right) + \left(2^{20} \cdot \frac{1}{2^{20}}\right) \\
&= 21
\end{aligned}$$

Under the reasonable assumption that you won't really get paid more than a million dollars, the expected payoff is a measley \$21!